出版前言

2011 年 12 月 11 日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰 100 周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共 10 卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

Preliminary Calculation of Circular Cylinder (Ⅱ)

PART (III)

Effect of an Elliptic Hole

Ref. Coker & Filon: Photo-elasticity pp 540-542

If we write

$$\chi_1 = e^{2\xi} + \cos 2\eta$$

$$\chi_2 = e^{-2\xi} + \cos 2\eta$$

$$\chi_3 = e^{-2\xi} \cos 2\eta$$

$$\chi_4 = \xi$$

$$\chi_5 = e^{2\xi} \cos 2\eta$$

then it is found that the stress function can be written as

$$\chi = \frac{1}{16} T \left\{ \chi_1 + (2e^{2\alpha} - 1)\chi_2 - e^{4\alpha}\chi_3 + 4(1 - \cosh 2\alpha)\chi_4 - \chi_5 \right\}$$

Now the stresses given by the different stress functions are, if we write $(\cosh 2\xi - \cos 2\eta) = 2J^2$,

$$\begin{cases} 2J^4\, \widehat{\xi\xi}_1 = \cos 4\eta - 4\cos 2\eta \cosh 2\xi + 2 + e^{4\xi} \\ 2J^4\, \widehat{\eta\eta}_1 = \cos 4\eta - 4\cos 2\eta\, e^{2\xi} + 2 + e^{4\xi} \\ 2J^4\, \widehat{\xi\eta}_1 = 2\sin 2\eta \cosh 2\xi \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_2} = \cos 4\eta - 4\cos 2\eta \cosh 2\xi + 2 + e^{-4\xi} \\ 2J^4\,\widehat{\eta\eta_2} = \cos 4\eta - 4\cos 2\eta \; e^{-2\xi} + 2 + e^{-4\xi} \\ 2J^4\,\widehat{\xi\eta_2} = -2\sin 2\eta \cosh 2\xi \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_3} = \cos 4\eta \cdot e^{-2\xi} - \cos 2\eta \left(e^{-4\xi}+3\right) + 3e^{-2\xi} \\ 2J^4\,\widehat{\eta\eta_3} = -\cos 4\eta \cdot e^{-2\xi} - 3e^{-2\xi} + \cos 2\eta \left(e^{-4\xi}+3\right) \\ 2J^4\,\widehat{\xi\eta_3} = \sin 4\eta \; e^{-2\xi} - \sin 2\eta \left(e^{-4\xi}+3\right) \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_4} = \sinh 2\xi \\ 2J^4\,\widehat{\eta\eta_4} = -\sinh 2\xi \\ 2J^4\,\widehat{\xi\eta_4} = \sin 2\eta \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_5} = \cos 4\eta \; e^{2\xi} - \cos 2\eta \left(e^{4\xi}+3\right) + 3e^{2\xi} \\ 2J^4\,\widehat{\eta\eta_5} = -\cos 4\eta \; e^{2\xi} + \cos 2\eta \left(e^{4\xi}+3\right) - 3e^{2\xi} \\ 2J^4\,\widehat{\eta\xi_5} = -\sin 4\eta \; e^{2\xi} + \sin 2\eta \left(e^{4\xi}+3\right) \end{cases}$$

To find the strain energy increase in the specimen, it is best to find the increase in work done by the external forces, because the difficulty of carrying out the integration in elliptical coordinates.

We have

$$\left.\begin{cases} 2\mu J u_1 = (2-4\sigma)\, e^{2\xi} - (4-4\sigma)\cos 2\eta \\ 2\mu J v_1 = -(2-4\sigma)\sin 2\eta \end{cases}\right\} \text{due to } X_1$$

$$\left.\begin{cases} 2\mu J u_2 = (4-4\sigma)\cos 2\eta - (2-4\sigma)\, e^{-2\xi} \\ 2\mu J v_2 = -(2-4\sigma)\sin 2\eta \end{cases}\right\} \text{due to } X_2$$

$$\left.\begin{cases} 2\mu J u_3 = 2\, e^{-2\xi}\cos 2\eta \\ 2\mu J v_3 = 2\, e^{-2\xi}\sin 2\eta \end{cases}\right\} \text{due to } X_3$$

$$\{\, 2\mu J v_4 = -1 \qquad \text{due to } X_4$$

$$\left.\begin{cases} 2\mu J v_5 = -2\, e^{2\xi}\cos 2\eta \\ 2\mu J v_5 = 2\, e^{2\xi}\sin 2\eta \end{cases}\right\} \text{due to } X_5$$

Therefore the total displacements

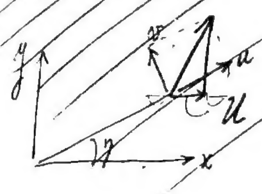
$$2\mu J u = \frac{1}{16} T \Big[(2-4\sigma) e^{2\xi} - (4-4\sigma)\cos 2\eta + (2e^{2\alpha}-1)\Big\{(4-4\sigma)\cos 2\eta - (2-4\sigma) e^{-2\xi}\Big\}$$
$$- e^{4\alpha}\, 2 e^{-2\xi} \cos 2\eta - 4(1-\cosh 2\alpha) + 2 e^{2\xi} \cos 2\eta \Big]$$

$$2\mu J v = \frac{1}{16} T \Big[-(2-4\sigma)\sin 2\eta - (2e^{2\alpha}-1)(2-4\sigma)\sin 2\eta - 2 e^{4\alpha} e^{-2\xi} \sin 2\eta - 2e^{2\xi}\sin 2\eta \Big]$$

or

$$\begin{cases} 2\mu J u = \dfrac{T}{8} \Big[(1-2\sigma) e^{2\xi} - (2-2\sigma)\cos 2\eta + (2e^{2\alpha}-1)\big\{(2-2\sigma)\cos 2\eta - (1-2\sigma)e^{-2\xi}\big\} \\ \qquad - e^{4\alpha} e^{-2\xi} \cos 2\eta - 2(1-\cosh 2\alpha) + e^{2\xi}\cos 2\eta \Big] \\ 2\mu J v = -\dfrac{T}{8} \sin 2\eta \Big[(1-2\sigma) + (2e^{2\alpha}-1)(1-2\sigma) + e^{4\alpha} e^{-2\xi} + e^{2\xi} \Big] \end{cases}$$

The component displacements in the direction of tension:



$\cancel{U = u\cos\eta - v\sin\eta}$

the uniform tension

$$\chi_0 = \frac{1}{2} T y^2 = \frac{T}{2} \sinh^2 \xi \sin^2 \eta$$

$$= \frac{T}{8} (\cosh 2\xi - 1)(1 - \cos 2\eta)$$

$$= \frac{T}{16} \left\{ (e^{2\xi} + \cos 2\eta) + (e^{-2\xi} + \cos 2\eta) - e^{2\xi} \cos 2\eta - e^{-2\xi} \cos 2\eta - 2 \right\}$$

$$= \frac{T}{16} \left\{ \chi_1 + \chi_2 - \chi_3 - \chi_5 \right\}$$

thus the displacements

$$2\mu J u_0 = \frac{T}{8} \left\{ 2(1-2\sigma) \sinh 2\xi + 2 \sinh 2\xi \cos 2\eta \right\}$$

$$2\mu J v_0 = -\frac{T}{8} \left\{ 2(1-2\sigma) \sin 2\eta + 2 \cosh 2\xi \sin 2\eta \right\}$$

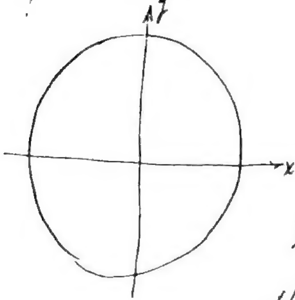
$$= -\frac{T}{8} \sin 2\eta \left\{ 2(1-2\sigma) + 2\cosh 2\xi \right\}$$

$$\cancel{T_\xi = T \cos \eta \qquad T_\eta = -T \sin \eta \quad \text{when } \xi = \infty}$$

$$2\mu J(u - u_0) = \frac{T}{8} \Big\{ (1-2\sigma) e^{-2\xi} - 2(1-\sigma) \cos 2\eta + (2e^{2\alpha} - 1)\{2(1-\sigma)\cos 2\eta - (1-2\sigma) e^{-2\xi}\}$$
$$- e^{4\alpha} e^{-2\xi} \cos 2\eta - 2(1 - \cosh 2\alpha) + e^{-2\xi} \cos 2\eta \Big\}$$

$$2\mu J(v - v_0) = -\frac{T}{8} \sin 2\eta \left\{ 2(e^{2\alpha} - 1) + e^{4\alpha} e^{-2\xi} - e^{-2\xi} \right\}$$

Now consider the circle at infinity



$$\sqrt{x^2+y^2} = \frac{c}{2} e^{\xi}$$

$$J^2 = \frac{1}{2}\frac{1}{2} e^{2\xi} c^2, \quad \therefore \quad J = \frac{c}{2} e^{\xi}$$

The work done by external forces will be the shear $\widehat{\xi\eta}$ + $\widehat{\xi\xi}$.

$$\widehat{\xi\eta} = -\frac{1}{2} T \sin 2\eta$$

$$\widehat{\xi\xi} = \frac{1}{2} T (1+\cos 2\eta)$$

Increase in strain energy

$$= \frac{T}{32\mu}\frac{T}{2}\left[\int_0^{2\pi} (1+\cos 2\eta)\left\{4(1-\sigma)(e^{2\alpha}-1)\cos 2\eta - 2(1-\cosh 2\alpha)\right\} d\eta + \int_0^{2\pi} \sin^2 2\eta \cdot 2(e^{2\alpha}-1)\, d\eta\right]$$

$$= \frac{T^2}{64\mu}\pi\left[4(1-\sigma)(e^{2\alpha}-1) + 4(\cosh 2\alpha - 1) + 2(e^{2\alpha}-1)\right]$$

$$= \frac{T^2}{32\mu}\pi\left[(3-2\sigma)(e^{2\alpha}-1) + 2(\cosh 2\alpha - 1)\right]$$

$$= \frac{T^2}{16\mu}\pi\left[(3-2\sigma)e^{\alpha}\sinh\alpha + (\cosh 2\alpha - 1)\right]c^2$$

Increase in strain energy $\mathcal{E}$

$$= \frac{(1+\sigma)T^2}{16E}\pi c^2\left[(3-2\sigma)(\sinh\alpha+\cosh\alpha)\sinh\alpha + (\cosh 2\alpha-1)\right]$$

The axis of the ellipse,

$$a = c\cosh\alpha \qquad a^2-b^2 = c^2$$

$$b = c\sinh\alpha$$

$$= t\frac{(1+\sigma)T^2}{16E}\pi\left[(3-2\sigma)(b+a)b + 2b^2\right]$$

$$\mathcal{E}_g = \frac{(1+\sigma)T^2\pi t}{16E}\left[(5-2\sigma)b^2 + (3-2\sigma)ab\right]$$

$$= \frac{(1+\sigma)T^2}{16E}(\pi ab)t\left[(3-2\sigma) + (5-2\sigma)\left(\frac{b}{a}\right)\right] \qquad \text{O.K.}$$

It is thus shown that the presence of a hole always increases the total strain energy, even compared with the whole plate plate. Therefore we have too much restraining, a buckling stress can only be arrived by considering more accurately the interaction.

Now for the sake of simplicity, go back to the case of a circular buckled region. Here, in order that the buckled circular plate be clamp supported, we choose the form of buckling to be

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}^2$$

Thus
$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial \theta} = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \\ \frac{1}{R}\frac{\partial w_0}{\partial \theta} = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial r \partial \theta} = \frac{1}{R}\frac{\partial^2 w_0}{\partial r \partial \theta} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin 2\theta$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{R}\frac{\partial^2 w_0}{\partial \theta^2} = -\left(\frac{r}{R}\right)^2 \cos 2\theta$$

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta + 4f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}\left(\frac{r}{R}\right)^2\frac{1}{r} \\ \frac{1}{R}\frac{\partial w_0}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta + 4f\left\{\left(\frac{a}{R}\right)^2 - 3\left(\frac{r}{R}\right)^2\right\}\frac{1}{R^2} \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta \end{cases}$$

$$\frac{1}{r^4}\left(\frac{\partial w}{\partial\theta}\right)^2 - \frac{1}{r^4}\left(\frac{\partial w_0}{\partial\theta}\right)^2 = 0$$

$$\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial r\,\partial\theta}\right)^2 - \frac{1}{r^2}\left(\frac{\partial^2 w_0}{\partial r\,\partial\theta}\right)^2 = 0$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}\left(\sin^2\theta\right)^2 - \frac{1}{R^2}\left[\sin^2\theta - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\right]\left[\sin^2\theta - 4f\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$= \frac{1}{R^2}\left[8f\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right)\sin^2\theta - 16f^2\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right\}$$

$$= \frac{1}{R^2}\cos 2\theta \cdot 4f\left\{\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right\}$$

$$\nabla^4\phi = \frac{4Ef}{R^2}\left[2\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right)\sin^2\theta + \cos 2\theta\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right) - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$= \frac{4Ef}{R^2}\left[\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right) - \left(\frac{r^2}{R^2}\right)\cos 2\theta - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

If we put $\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{R^2} - \dots$ [illegible]

$$\nabla^4 \varphi = C\left[p^2-4\right]\left[(p-2)^2-4\right] r^{p-4} \cos 2\theta$$

$p = 6,$ $$\nabla^4 \varphi = C \cdot 32 \cdot 12\, r^2 \cos 2\theta$$

$$\therefore \quad C = \frac{-K}{384}$$

$$\therefore \quad \varphi_4 = -\frac{Ef}{384 R^2} \frac{r^6}{R^2} \cos 2\theta$$

Therefore the particular integral is

$$\boxed{\varphi_0 = \frac{EfR^2}{64}\left[\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\frac{r^4}{R^4} + \frac{2}{9}\left(2f\frac{a^2}{R^2} - 1\right)\frac{r^6}{R^6} - \frac{1}{6}\frac{r^6}{R^6}\cos 2\theta - \frac{1}{12} f \frac{r^8}{R^8}\right]}$$

Due to the symmetry of this problem, the solution of the homogeneous equation

$$\nabla^4 \varphi = 0$$

can be written as

$$\frac{\varphi_1}{R^2} = \left(\frac{Ef}{64}\right)\left[\frac{1}{4} Q_0 \frac{r^2}{R^2} + S_0 + \cos 2\theta\left[P_2\left(\frac{r}{R}\right)^2 + R_2\left(\frac{r}{R}\right)^4 \cdots\right] + \cancel{\cos 4\theta\left[P_4\left(\frac{r}{R}\right)^4 + R_4\left(\frac{r}{R}\right)^6\right]} + \cos 6\theta\left[P_6\left(\frac{r}{R}\right)^6 + R_6\left(\frac{r}{R}\right)^8\right]\right]$$

$$\frac{\varphi}{R^2} = \left(\frac{Ef}{64}\right)\Bigg[\Big\{ \cancel{Q} + \frac{1}{4}Q_0\left(\frac{r}{R}\right)^2 + \left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^4 + \frac{2}{9}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^6$$

$$- \frac{1}{12} f \left(\frac{r}{R}\right)^8 \Big\}$$

$$+ \cos 2\theta \left\{ P_2\left(\frac{r}{R}\right)^2 + R_2\left(\frac{r}{R}\right)^4 + \quad - \frac{1}{6}\left(\frac{r}{R}\right)^6 \right\}$$

$$\cancel{+ \cos 4\theta \left\{ P_4\left(\frac{r}{R}\right)^4 + R_4\left(\frac{r}{R}\right)^6 \right\}}$$

$$\cancel{+ \cos 6\theta \left\{ P_6\left(\frac{r}{R}\right)^6 + R_6\left(\frac{r}{R}\right)^8 \right\}}$$

$$\frac{1}{r}\frac{\partial\varphi}{\partial r} = \left(\frac{Ef}{64}\right)\Bigg[\Big\{ \frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6 \Big\}$$

$$+ \cos 2\theta \left\{ 2P_2 + 4R_2\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4 \right\}$$

$$\cancel{+ \cos 4\theta \left\{ 4P_4\left(\frac{r}{R}\right)^2 + 6R_4\left(\frac{r}{R}\right)^4 \right\}}$$

$$\cancel{+ \cos 6\theta \left\{ 6P_6\left(\frac{r}{R}\right)^4 + 8R_6\left(\frac{r}{R}\right)^6 \right\}}$$

$$\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = \frac{Ef}{64}\Bigg[-4\cos 2\theta \left\{ P_2 + R_2\left(\frac{r}{R}\right)^2 - \frac{1}{6}\left(\frac{r}{R}\right)^4 \right\}$$

$$\cancel{- 16\cos 4\theta \left\{ P_4\left(\frac{r}{R}\right)^2 + R_4\left(\frac{r}{R}\right)^4 \right\}}$$

$$\cancel{- 36\cos 6\theta \left\{ P_6\left(\frac{r}{R}\right)^4 + R_6\left(\frac{r}{R}\right)^6 \right\} \Bigg]}$$

$$\widehat{rr} = \frac{Ef}{64}\Big[\Big\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^2\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6\Big\}$$

$$-\cos 2\theta\Big\{2P_2 + \frac{1}{3}\left(\frac{r}{R}\right)^4\Big\}$$

$$-\cos 4\theta\Big\{12P_4\left(\frac{r}{R}\right)^2 + 10R_4\left(\frac{r}{R}\right)^4\Big\}$$

$$-\cos 6\theta\Big\{30P_6\left(\frac{r}{R}\right)^4 + 28R_6\left(\frac{r}{R}\right)^6\Big\}\Big]$$

$$\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{20}{3}\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{14}{3}f\left(\frac{r}{R}\right)^6\Big\}$$

$$+\cos 2\theta\Big\{P_2 + 12R_2\left(\frac{r}{R}\right)^2 - 5\left(\frac{r}{R}\right)^4\Big\} \qquad !!!$$

$$+\cos 4\theta\Big\{12P_4\left(\frac{r}{R}\right)^2 + 30R_4\left(\frac{r}{R}\right)^4\Big\}$$

$$+\cos 6\theta\Big\{30P_6\left(\frac{r}{R}\right)^4 + 56R_6\left(\frac{r}{R}\right)^6\Big\}$$

$$\widehat{r\theta} = \left(\frac{Ef}{64}\right)\Big[2\sin 2\theta\Big\{P_2 + 3R_2\left(\frac{r}{R}\right)^2 - \frac{5}{6}\left(\frac{r}{R}\right)^4\Big\}$$

$$+4\sin 3\theta\Big\{3P_4\left(\frac{r}{R}\right)^2 + 5R_4\left(\frac{r}{R}\right)^4\Big\}$$

$$+6\sin 6\theta\Big\{5P_6\left(\frac{r}{R}\right)^4 + 7R_6\left(\frac{r}{R}\right)^6\Big\}\Big]$$

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left(\widehat{rr} - \nu\,\widehat{\theta\theta}\right)$$

$$\therefore\ \frac{u}{R} = \frac{p}{64}\Bigg[\Big\{(1-\nu)\frac{1}{2}Q_0\left(\frac{r}{R}\right) + \frac{4}{3}(1-3\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^3$$

$$+\frac{4}{15}(1-5\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^5 - \frac{2}{21}(1-7\nu)f\left(\frac{r}{R}\right)^7\Big\}$$

$$-\cos 2\theta\Big\{(2+\nu)P_2\left(\frac{r}{R}\right) + 4\nu R_2\left(\frac{r}{R}\right)^3 + \frac{1}{15}(1-15\nu)\left(\frac{r}{R}\right)^5\Big\}$$

$$-\cos 4\theta\Big\{4(1+\nu)P_4\left(\frac{r}{R}\right)^3 + 2(1+3\nu)R_4\left(\frac{r}{R}\right)^5\Big\}$$

$$-\cos 6\theta\Big\{6(1+\nu)P_6\left(\frac{r}{R}\right)^5 + 4(1+2\nu)R_6\left(\frac{r}{R}\right)^7\Big\} + F(\theta)$$

$$\frac{1}{E}\left(\widehat{\theta\theta} - \nu\,\widehat{rr}\right) = \frac{p}{64}\Bigg[\Big\{(1-\nu)\frac{Q_0}{2} + 4(3-\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2$$

$$+\frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}(7-\nu)f\left(\frac{r}{R}\right)^6\Big\}$$

$$+\cos 2\theta\Big\{(1+2\nu)P_2 + 12R_2\left(\frac{r}{R}\right)^2 - \left(5-\frac{1}{3}\nu\right)\left(\frac{r}{R}\right)^4\Big\}$$

$$+\cos 4\theta\Big\{12(1+\nu)P_4\left(\frac{r}{R}\right)^2 + 10(3+\nu)R_4\left(\frac{r}{R}\right)^4\Big\}$$

$$+\cos 6\theta\Big\{30(1+\nu)P_6\left(\frac{r}{R}\right)^4 + 28(2+\nu)R_6\left(\frac{r}{R}\right)^6\Big\}$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2-\left(\frac{\partial w_0}{\partial r}\right)^2\right\}+\frac{\partial u}{\partial r}=\frac{1}{E}\left(\widehat{rr}-\nu\widehat{\theta\theta}\right)$$

here

$$\frac{1}{E}\left(\widehat{rr}-\nu\widehat{\theta\theta}\right)=\frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2}+4(1-3\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2$$

$$+\frac{4}{3}(1-5\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4-\frac{2}{3}(1-7\nu)f^2\left(\frac{r}{R}\right)^6\Big\}$$

$$-\cos 2\theta\left\{(2+\nu)P_2+12\nu R_2\left(\frac{r}{R}\right)^2-\left(\frac{1}{3}-5\nu\right)\left(\frac{r}{R}\right)^4\right\}$$

$$-\cos 4\theta\left\{12(1+\nu)P_4\left(\frac{r}{R}\right)^2+10(1+3\nu)R_4\left(\frac{r}{R}\right)^4\right\}$$

$$-\cos 6\theta\left\{30(1+\nu)P_6\left(\frac{r}{R}\right)^4+98(1+2\nu)R_6\left(\frac{r}{R}\right)^6\right\}$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2-\left(\frac{\partial w_0}{\partial r}\right)^2\right\}=\frac{1}{2}\left[\left\{\left(\frac{r}{R}\right)\sin^2\theta-4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\frac{r}{R}\right\}^2-\left\{\frac{r}{R}\sin^2\theta\right\}^2\right]$$

$$=\frac{1}{2}\left(\frac{r}{R}\right)^2 4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\left[4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)-2\sin^2\theta\right] \quad * \ 4f=f$$

$$=\frac{f}{2}\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\frac{r^2}{R^2}\left[f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)+\cos 2\theta-1\right]$$

$$=\frac{f}{64}\Big[\Big\{32\frac{a^2}{R^2}\left(f\frac{a^2}{R^2}-1\right)\frac{r^2}{R^2}-32\left(2f\frac{a^2}{R^2}-1\right)\frac{r^4}{R^4}+32f\left(\frac{r}{R}\right)^6\Big\}$$

$$+\cos 2\theta\left\{32\frac{a^2}{R^2}\frac{r^2}{R^2}-32\frac{r^4}{R^4}\right\}\Big]$$

$$\frac{\partial u}{\partial r} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2} + 12(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2$$

$$+ \frac{20}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{14}{3}(7-\nu) f \left(\frac{r}{R}\right)^6\Big\}$$

$$- \cos 2\theta\Big\{(2+\nu)P_2 + 4\left(8\frac{a^2}{R^2} + 3\nu R_2\right)\left(\frac{r}{R}\right)^2 - \left(\frac{97}{3} - 5\nu\right)\left(\frac{r}{R}\right)^4\Big\}$$

$$- \cos 4\theta\Big\{12(1+\nu)P_4\left(\frac{r}{R}\right)^2 + 10(1+5\nu)R_4\left(\frac{r}{R}\right)^4\Big\}$$

$$- \cos 6\theta\Big\{30(1+\nu)P_6\left(\frac{r}{R}\right)^4 + 28(1+9\nu)R_6\left(\frac{r}{R}\right)^6\Big\}$$

$$\frac{u}{R} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2}\left(\frac{r}{R}\right) + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^3 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^5$$

$$- \frac{2}{3}(7-\nu) f \left(\frac{r}{R}\right)^7\Big\}$$

$$- \cos 2\theta\Big\{(2+\nu)P_2\left(\frac{r}{R}\right) + \frac{4}{3}\left(8\frac{a^2}{R^2} + 3\nu R_2\right)\left(\frac{r}{R}\right)^3 - \left(\frac{97}{15} - \nu\right)\left(\frac{r}{R}\right)^5\Big\}$$

$$- \cos 4\theta\Big\{4(1+\nu)P_4\left(\frac{r}{R}\right)^3 + 2(1+3\nu)R_4\left(\frac{r}{R}\right)^5\Big\}$$

$$- \cos 6\theta\Big\{6(1+\nu)P_6\left(\frac{r}{R}\right)^5 + 4(1+9\nu)R_6\left(\frac{r}{R}\right)^7\Big\}\Big] + F(\theta)$$

$\therefore$

$$\frac{v}{R} = \frac{p}{64}\Big[\sin 2\theta\Big\{\tfrac{3}{2}(1+\nu)P_2\left(\tfrac{r}{R}\right) + 2\left(\tfrac{8}{3}\tfrac{a^2}{R^2} + \overline{3+\nu}\,Q_2\right)\left(\tfrac{r}{R}\right)^3 - \left(\tfrac{86}{15} - \tfrac{2}{3}\nu\right)\left(\tfrac{r}{R}\right)^5\Big\}$$

$$+ \sin 4\theta\Big\{4(1+\nu)P_4\left(\tfrac{r}{R}\right)^3 + 2(4+2\nu)R_4\left(\tfrac{r}{R}\right)^5\Big\}$$

$$+ \sin 6\theta\Big\{6(1+\nu)P_6\left(\tfrac{r}{R}\right)^5 + 2(5+3\nu)R_6\left(\tfrac{r}{R}\right)^7\Big\}\Big] - \int F(\theta)\,d\theta + G\left(\tfrac{r}{R}\right)$$

$$\frac{\int F(\theta)d\theta}{\left(\frac{r}{R}\right)} + \frac{F'(\theta)}{\left(\frac{r}{R}\right)} + G'\left(\tfrac{r}{R}\right) - \frac{G\left(\frac{r}{R}\right)}{\left(\frac{r}{R}\right)} = 0$$

or $\int F(\theta)d\theta + F'(\theta) = G\left(\tfrac{r}{R}\right) - \tfrac{r}{R}G'\left(\tfrac{r}{R}\right)$

or $\int F(\theta)d\theta + F'(\theta) = C$

$G\left(\tfrac{r}{R}\right) - \tfrac{r}{R}G'\left(\tfrac{r}{R}\right) = C$ $\qquad \therefore F \cdot G = 0$

or $F(\theta) + F''(\theta) = 0$

$F'' + F = 0$ only $F = A\,{\sin\theta \atop \cos\theta} > \text{cut}$

$$G'\left(\tfrac{r}{R}\right) - \frac{1}{\left(\frac{r}{R}\right)}G\left(\tfrac{r}{R}\right) = -\frac{C}{\left(\frac{r}{R}\right)}$$

$$\left(\tfrac{r}{R}\right)\frac{d}{d\left(\frac{r}{R}\right)}\left[\frac{1}{\left(\frac{r}{R}\right)}G\left(\tfrac{r}{R}\right)\right] = -\frac{C}{\left(\frac{r}{R}\right)} \quad \Big\| \quad \frac{1}{\left(\frac{r}{R}\right)}G = \frac{C}{\left(\frac{r}{R}\right)} + B$$

$$G = C + B\left(\tfrac{r}{R}\right)$$

The undisturbed stress function outside the circular region

$$\frac{\phi_1}{R^2} = \tfrac{1}{4}\sigma\,(1-\cos 2\theta)\left(\frac{r}{R}\right)^2$$

The other possible solutions are

$$\frac{\phi_2}{R^2} = \sigma\Big[\,P_0 \log\left(\frac{r}{R}\right) + \cos 2\theta\left\{Q_2\frac{1}{\left(\frac{r}{R}\right)^2} + S_2\right\}$$
$$+\cos 4\theta\left\{Q_4\frac{1}{\left(\frac{r}{R}\right)^4} + S_4\frac{1}{\left(\frac{r}{R}\right)^2}\right\}$$
$$+\cos 6\theta\left\{Q_6\frac{1}{\left(\frac{r}{R}\right)^6} + S_6\frac{1}{\left(\frac{r}{R}\right)^4}\right\}\Big]$$

$$\frac{1}{\left(\frac{r}{R}\right)}\frac{\partial\left(\frac{\phi_2}{R^2}\right)}{\partial\left(\frac{r}{R}\right)} = \sigma\Big[\,P_0\frac{1}{\left(\frac{r}{R}\right)^2} + \cos 2\theta\left\{-\frac{2Q_2}{\left(\frac{r}{R}\right)^4}\right\}$$
$$+\cos 4\theta\left\{-\frac{4Q_4}{\left(\frac{r}{R}\right)^6} - \frac{2S_4}{\left(\frac{r}{R}\right)^4}\right\}$$
$$+\cos 6\theta\left\{-\frac{6Q_6}{\left(\frac{r}{R}\right)^8} - \frac{4S_6}{\left(\frac{r}{R}\right)^6}\right\}\Big]$$

$$\frac{1}{\left(\frac{r}{R}\right)^2}\frac{\partial^2\left(\frac{\phi_2}{R^2}\right)}{\partial\theta^2} = \sigma\Big[-4\cos 2\theta\left\{\frac{Q_2}{\left(\frac{r}{R}\right)^4} + \frac{S_2}{\left(\frac{r}{R}\right)^2}\right\} - 16\cos 4\theta\left\{\frac{Q_4}{\left(\frac{r}{R}\right)^6} + \frac{S_4}{\left(\frac{r}{R}\right)^4}\right\}$$
$$-36\cos 6\theta\left\{\frac{Q_6}{\left(\frac{r}{R}\right)^8} + \frac{S_6}{\left(\frac{r}{R}\right)^6}\right\}\Big]$$

$$\widehat{rr}_1 = \sigma\Bigg[\frac{R_0}{(\frac{r}{R})^2} - \cos 2\theta\left\{\frac{6Q_2}{(\frac{r}{R})^4} + \frac{4S_2}{(\frac{r}{R})^2}\right\}$$

$$- \cos 4\theta\left\{\frac{20Q_4}{(\frac{r}{R})^6} + \frac{18S_4}{(\frac{r}{R})^4}\right\} - \cos 6\theta\left\{\frac{42Q_6}{(\frac{r}{R})^8} + \frac{40S_6}{(\frac{r}{R})^6}\right\}\Bigg]$$

$$\widehat{\theta\theta}_1 = \sigma\Bigg[-\frac{R_0}{(\frac{r}{R})^2} + \cos 2\theta \cdot \frac{6Q_2}{(\frac{r}{R})^4} + \cos 4\theta\left\{\frac{20Q_4}{(\frac{r}{R})^6} + \frac{6S_4}{(\frac{r}{R})^4}\right\}$$

$$+ \cos 6\theta\left\{\frac{42Q_6}{(\frac{r}{R})^8} + \frac{20S_6}{(\frac{r}{R})^6}\right\}\Bigg]$$

$$\widehat{r\theta}_2 = \sigma\Bigg[-2\sin 2\theta\left\{\frac{3Q_2}{(\frac{r}{R})^4} + \frac{S_2}{(\frac{r}{R})^2}\right\} - 4\sin 4\theta\left\{\frac{5Q_4}{(\frac{r}{R})^6} + \frac{3S_4}{(\frac{r}{R})^4}\right\}$$

$$- 6\sin 6\theta\left\{\frac{7Q_6}{(\frac{r}{R})^8} + \frac{5S_6}{(\frac{r}{R})^6}\right\}\Bigg]$$

Therefore $$\widehat{rr} = \sigma\Bigg[\frac{1}{2} + \frac{R_0}{(\frac{r}{R})^2} + \cos 2\theta\left\{\frac{1}{2} - \frac{6Q_2}{(\frac{r}{R})^4} - \frac{4S_2}{(\frac{r}{R})^2}\right\}$$

$$- \cancel{\cos 4\theta\left\{\frac{20Q_4}{(\frac{r}{R})^6} + \frac{18S_4}{(\frac{r}{R})^4}\right\} - \cos 6\theta\left\{\frac{42Q_6}{(\frac{r}{R})^8} + \frac{40S_6}{(\frac{r}{R})^6}\right\}}\Bigg]$$

$$\widehat{\theta\theta} = \sigma\Bigg[\frac{1}{2} - \frac{R_0}{(\frac{r}{R})^2} + \cos 2\theta\left\{\frac{6Q_2}{(\frac{r}{R})^4} - \frac{1}{2}\right\} + \cancel{\cos 4\theta\left\{\frac{20Q_4}{(\frac{r}{R})^6} + \frac{6S_4}{(\frac{r}{R})^4}\right\}}$$

$$+ \cos 6\theta\left\{\frac{42Q_6}{(\frac{r}{R})^8} + \frac{30S_6}{(\frac{r}{R})^6}\right\}\Bigg]$$

$$\widehat{r\theta} = -\sigma\left[\sin 2\theta\left\{\frac{1}{2} + \frac{6Q_2}{\left(\frac{a}{R}\right)^4} + \frac{2S_2}{\left(\frac{a}{R}\right)^2}\right\} + \sin 4\theta\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{12S_4}{\left(\frac{a}{R}\right)^4}\right\}\right.$$
$$\left. + \sin 6\theta\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{30S_6}{\left(\frac{a}{R}\right)^6}\right\}\right]$$

The stress conditions at the boundary of the circular region are then

$$\sigma\left\{\frac{1}{2} + \frac{R_0}{\left(\frac{a}{R}\right)^2}\right\} = \frac{Ef}{64}\left\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^4\left(1 - f\frac{a^2}{R^2}\right) + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4 - \frac{2}{3}f\left(\frac{a}{R}\right)^6\right\} \quad (1)$$

$$\sigma\left\{\frac{6Q_2}{\left(\frac{a}{R}\right)^4} + \frac{4S_2}{\left(\frac{a}{R}\right)^2} - \frac{1}{2}\right\} = \frac{Ef}{64}\left\{2P_2 + \frac{1}{3}\left(\frac{a}{R}\right)^4\right\} \quad (2)$$

$$\sigma\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{18S_4}{\left(\frac{a}{R}\right)^4}\right\} = \frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2 + 10R_4\left(\frac{a}{R}\right)^4\right\} \quad (3)$$

$$\sigma\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{40S_6}{\left(\frac{a}{R}\right)^6}\right\} = \frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4 + 28R_6\left(\frac{a}{R}\right)^6\right\} \quad (4)$$

$$\sigma\left\{\frac{1}{2} - \frac{R_0}{\left(\frac{a}{R}\right)^2}\right\} = \frac{Ef}{64}\left\{\frac{1}{2}Q_0 + 12\left(\frac{a}{R}\right)^4\left(1 - f\frac{a^2}{R^2}\right) + \frac{20}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^2 - \frac{17}{3}f\left(\frac{a}{R}\right)^6\right\} \quad (5)$$

$$\sigma\left\{\frac{6Q_2}{\left(\frac{a}{R}\right)^4} - \frac{1}{2}\right\} = \frac{Ef}{64}\left\{P_2 + 12R_2\left(\frac{a}{R}\right)^2 - 5\left(\frac{a}{R}\right)^4\right\} \quad (6)$$

$$\sigma\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{6S_4}{\left(\frac{a}{R}\right)^4}\right\} = \frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2 + 30R_4\left(\frac{a}{R}\right)^4\right\} \quad (7)$$

$$\sigma\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{90S_6}{\left(\frac{a}{R}\right)^6}\right\} = \frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4 + 56R_6\left(\frac{a}{R}\right)^6\right\} \quad (8)$$

$$-\sigma\left\{\frac{1}{2}+\frac{6Q_2}{\left(\frac{a}{R}\right)^4}+\frac{2S_2}{\left(\frac{a}{R}\right)^2}\right\}=\frac{Et}{64}\left\{2P_2+6P_2\left(\frac{a}{R}\right)^2-\frac{5}{3}\left(\frac{a}{R}\right)^4\right\} \qquad (9)$$

$$-\sigma\left\{\frac{30\,Q_4}{\left(\frac{a}{R}\right)^6}+\frac{12\,S_4}{\left(\frac{a}{R}\right)^4}\right\}=\frac{Et}{64}\left\{12\,P_4\left(\frac{a}{R}\right)^2+20\,P_4\left(\frac{a}{R}\right)^4\right\} \qquad (10)$$

$$-\sigma\left\{\frac{42\,Q_6}{\left(\frac{a}{R}\right)^8}+\frac{30\,S_6}{\left(\frac{a}{R}\right)^6}\right\}=\frac{Et}{64}\left\{30\,P_6\left(\frac{a}{R}\right)^4+42\,P_6\left(\frac{a}{R}\right)^6\right\} \qquad (11)$$

$$\frac{u}{R}=\frac{\sigma}{E}\left[\frac{1}{2}(1-\nu)\left(\frac{a}{R}\right)-(1+\nu)P_0\frac{1}{\left(\frac{a}{R}\right)}+\cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{a}{R}\right)+2Q_2(1+\nu)\frac{1}{\left(\frac{a}{R}\right)^3}+\frac{4S_2}{\left(\frac{a}{R}\right)}\right\}\right.$$
$$\left.+\cos 4\theta\left\{4(1+\nu)\frac{Q_4}{\left(\frac{a}{R}\right)^5}+2(3+\nu)\frac{S_4}{\left(\frac{a}{R}\right)^3}\right\}+\cos 6\theta\left\{6(1+\nu)\frac{Q_6}{\left(\frac{a}{R}\right)^7}+4(2+\nu)\frac{S_6}{\left(\frac{a}{R}\right)^5}\right\}\right]$$

$$\frac{1}{E}\left(\widehat{\theta\theta}-\nu\widehat{rr}\right)=\frac{\sigma}{E}\left[\frac{1}{2}(1-\nu)-(1+\nu)P_0\frac{1}{\left(\frac{a}{R}\right)^2}+\cos 2\theta\left\{-\frac{1}{2}(1+\nu)+\frac{6(1+\nu)Q_2}{\left(\frac{a}{R}\right)^4}+\frac{4S_2}{\left(\frac{a}{R}\right)^2}\right\}\right.$$
$$\left.+\cos 4\theta\left\{\frac{20(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6}+\frac{6(1+3\nu)S_4}{\left(\frac{a}{R}\right)^4}\right\}+\cos 6\theta\left\{\frac{42(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8}+\frac{30(1+2\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\}\right]$$

$$\frac{v}{R}=\frac{\sigma}{E}\left[\sin 2\theta\left\{\frac{2(1+\nu)Q_2}{\left(\frac{a}{R}\right)^3}-\frac{1}{2}(1+\nu)\left(\frac{a}{R}\right)\right\}\right.$$
$$\left.+\sin 4\theta\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^5}+\frac{4\nu\,S_4}{\left(\frac{a}{R}\right)^3}\right\}+\sin 6\theta\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^7}+\frac{(2+6\nu)S_6}{\left(\frac{a}{R}\right)^5}\right\}\right]$$

The boundary conditions at the periphery of the circular region is then

$$* \quad \sigma\left\{\frac{1}{2}(1-\nu)-(1+\nu)\frac{P_0}{\left(\frac{a}{R}\right)^2}\right\}=\frac{Ef}{64}\left\{(1-\nu)\frac{Q_0}{2}+4(3-\nu)\left(\frac{a}{R}\right)^4\left(1-f\frac{a^2}{R^2}\right)+\frac{4}{3}(5-\nu)\left(2+f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^4\right.$$

$(5-\nu)(\ )$

$$\left.-\frac{2}{3}(7-\nu)f\left(\frac{a}{R}\right)^6\right\} \qquad (12)$$

$$-\sigma\left\{\frac{1+\nu}{2}+\frac{2Q_2(1+\nu)}{\left(\frac{a}{R}\right)^4}+\frac{4S_2}{\left(\frac{a}{R}\right)^2}\right\}=\frac{Ef}{64}\left\{(2+\nu)P_2+\frac{4}{3}\left(f\frac{a^2}{R^2}+3\nu R_2\right)\left(\frac{a}{R}\right)^2-\left(\frac{97}{15}-\nu\right)\left(\frac{a}{R}\right)^4\right\} \qquad (13)$$

$$-\sigma\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6}+\frac{2(3+\nu)S_4}{\left(\frac{a}{R}\right)^4}\right\}=\frac{Ef}{64}\left\{4(1+\nu)P_4\left(\frac{a}{R}\right)^2+2(1+3\nu)R_4\left(\frac{a}{R}\right)^4\right\} \qquad (14)$$

$$-\sigma\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8}+\frac{4(2+\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\}=\frac{Ef}{64}\left\{6(1+\nu)P_6\left(\frac{a}{R}\right)^4+4(1+2\nu)R_6\left(\frac{a}{R}\right)^6\right\} \qquad (15)$$

$$\sigma\left\{\frac{2(1+\nu)S_2}{\left(\frac{a}{R}\right)^4}-\frac{1}{2}(1+\nu)\right\}=\frac{Ef}{64}\left\{\frac{3}{2}(1+\nu)P_2+2\left(\frac{f}{3}\frac{a^2}{R^2}+3+\nu R_2\right)\left(\frac{a}{R}\right)^2-\left(\frac{86}{15}-\frac{2}{3}\nu\right)\left(\frac{a}{R}\right)^4\right\} \qquad (16)$$

$$\sigma\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6}+\frac{4\nu S_4}{\left(\frac{a}{R}\right)^4}\right\}=\frac{Ef}{64}\left\{4(1+\nu)P_4\left(\frac{a}{R}\right)^2+2(4+2\nu)R_4\left(\frac{a}{R}\right)^4\right\} \qquad (17)$$

$$\sigma\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8}+\frac{2(1+3\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\}=\frac{Ef}{64}\left\{6(1+\nu)P_6\left(\frac{a}{R}\right)^4+2(5+3\nu)R_6\left(\frac{a}{R}\right)^6\right\} \qquad (18)$$

Let us investigate the equations (2), (6), (9), (13) & (16)

$$6q_2 + 4s_2 - \frac{1}{2} = \left\{ 3p_2' + \frac{1}{3}h \right\} \qquad h = \left(\frac{a}{R}\right)^2$$

$$6q_2 - \frac{1}{2} = -\left\{ p_2' + 12r_2' - 5h' \right\}$$

$$-\left(6q_2 + 2s_2 + \frac{1}{2}\right) = \left\{ +2p_2' + 6r_2' - \frac{5}{3}h' \right\}$$

$$-\left[\frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2\right] = \left\{ (2+\nu)p_2' + 4\nu r_2' + \left(\frac{21}{5}+\nu\right)h \right\}$$

$$\left[2(1+\nu)q_2 - \frac{1}{2}(1+\nu)\right] = \left\{ \frac{3}{2}(1+\nu)p_2' + 2(3+\nu)r_2' - \left(\frac{2}{5} - \frac{2}{3}\nu\right)h' \right\}$$

The question is whether this system of equations are consistent. They are not consistent, so we can only satisfy them approximately, by means of method of least square, thus

$$6\left(6q_2 + 4s_2 - \frac{1}{2} - 3p_2 - \frac{1}{3}h\right) + 6\left(6q_2 - \frac{1}{2} - p_2 - 12r_2 + 5h\right)$$

$$+ 6\left(6q_2 + 2s_2 + \frac{1}{2} + 2p_2 + 6r_2 - \frac{5}{3}h\right)$$

$$+ 2(1+\nu)\left[\frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2 + (2+\nu)p_2 + 4\nu r_2 + \left(\frac{21}{5}+\nu\right)h\right]$$

$$+ 2(1+\nu)\left[2(1+\nu)q_2 - \frac{1}{2}(1+\nu) - \frac{3}{2}(1+\nu)p_2 - 2(3+\nu)r_2 + \left(\frac{2}{5} - \frac{2}{3}\nu\right)h\right] = 0.$$

or

$$\left[108 + 4(1+\nu)^2\right]q_2 + \left[36 + 8(1+\nu)\right]s_2 + \left[(1-\nu^2) - 6\right]p_2 + \left[4(1+\nu)(-3+\nu) - 36\right]r_2 + \left[-3 + 18h + 2(1+\nu)\left(\frac{23}{5} + \frac{1}{3}\right)h\right] = 0 \qquad (A)$$

$$4\left(6q_2+4s_2-\frac{1}{2}-2p_2-\frac{h}{3}\right)+2\left(6q_2+2s_2+\frac{1}{2}+2p_2+6r_2-\frac{5}{3}h\right)$$

$$+4\left[\frac{1+\nu}{2}+2(1+\nu)q_2+4s_2+(2+\nu)p_2+4\nu r_2+\left(\frac{21}{5}+\nu\right)h\right]=0$$

$$\boxed{[36+8(1+\nu)]q_2+36s_2+4(1+\nu)p_2+4(3+4\nu)r_2+\left[1+2\nu+\left(\frac{112}{15}+4\nu\right)h\right]=0}$$

(B)

$$2\left(2p_2+\frac{h}{3}-6q_2-4s_2+\frac{1}{2}\right)+\left(p_2+12r_2-5h-6q_2+\frac{1}{2}\right)$$

$$+2\left(2p_2+6r_2-\frac{5}{3}h+6q_2+2s_2+\frac{1}{2}\right)+(2+\nu)\left[(2+\nu)p_2+4\nu r_2+\left(\frac{21}{5}+\nu\right)h\right.$$

$$\left.+\frac{1+\nu}{2}+2(1+\nu)q_2+4s_2\right]+\frac{3}{2}(1+\nu)\left[\frac{3}{2}(1+\nu)p_2+2(3+\nu)r_2-\left(\frac{2}{5}-\frac{2}{3}\nu\right)h\right.$$

$$\left.-2(1+\nu)q_2+\frac{1}{2}(1+\nu)\right]=0.$$

$$[-6+(1-\nu^2)]q_2+4(1+\nu)s_2+\left[9+(2+\nu)^2+\frac{9}{4}(1+\nu)^2\right]p_2+\left[24+4(2+\nu)\nu+3(1+\nu)(3+\nu)\right]r_2$$

$$+\left[\frac{5}{2}+\frac{1}{2}(1+\nu)\left(\frac{7}{2}+\frac{5}{2}\nu\right)+h\left\{\frac{2}{3}-5-\frac{10}{3}+(2+\nu)\left(\frac{21}{5}+\nu\right)-\frac{3}{2}(1+\nu)\left(\frac{2}{5}-\frac{2}{3}\nu\right)\right\}\right]=0$$

(C)

$$6\left(p_2 + 12 r_2 - 5h - 6q_2 + \tfrac{1}{2}\right) + 3\left(2p_2 + 6r_2 - \tfrac{5}{3}h + 6q_2 + 9s_2 + \tfrac{1}{2}\right)$$

$$+ 2\nu\left[(2+\nu)p_2 + 4\nu r_2 + \left(\tfrac{21}{5}+\nu\right)h + \tfrac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2\right]$$

$$+ (3+\nu)\left[\tfrac{3}{2}(1+\nu)p_2 + 2(3+\nu)r_2 - \left(\tfrac{2}{5} - \tfrac{2}{3}\nu\right)h - 2(1+\nu)q_2 + \tfrac{1}{2}(1+\nu)\right] = 0.$$

$$\boxed{\begin{aligned} &\left[-18 + 4\nu(1+\nu) - 2(1+\nu)(3+\nu)\right]q_2 + \left[6 + 8\nu\right]s_2 \\ &+ \left[12 + 2\nu(2+\nu) + \tfrac{3}{2}(1+\nu)(3+\nu)\right]p_2 + \left[90 + 8\nu^2 + 2(3+\nu)^2\right]r_2 \\ &+ \left[\tfrac{9}{2} + \nu(1+\nu) + \tfrac{1}{2}(1+\nu)(3+\nu) + h\left\{-35 + 2\nu\left(\tfrac{21}{5}+\nu\right) - 2(3+\nu)\left(\tfrac{1}{5} - \tfrac{1}{3}\nu\right)\right\}\right] = 0 \qquad \text{(D)} \end{aligned}}$$

The equations (A), (B), (C), (D) determines $\boxed{p_2,\ q_2,\ r_2,\ s_2}$

$$\widehat{rr}+\widehat{\theta\theta}=\sigma\left[1+\cos 2\theta\left\{-\frac{4S_2}{\left(\frac{r}{R}\right)^2}\right\}\right]$$

$$\int_0^{2\pi} d\theta\left\{\left(\widehat{rr}+\widehat{\theta\theta}\right)^2-2(1+\nu)\left(\widehat{rr}\cdot\widehat{\theta\theta}-\widehat{r\theta}^2\right)\right\}$$

$$=\pi\sigma^2\left[2+\frac{16S_2^2}{\left(\frac{r}{R}\right)^4}-2(1+\nu)\left\{\frac{1}{2}-\frac{2R_0^2}{\left(\frac{r}{R}\right)^4}-\left(\frac{1}{2}-\frac{6Q_2}{\left(\frac{r}{R}\right)^4}\right)^2+\frac{4S_2}{\left(\frac{r}{R}\right)^2}\left(\frac{1}{2}-\frac{6Q_2}{\left(\frac{r}{R}\right)^4}\right)\right.\right.$$

$$\left.\left.-\left(\frac{1}{2}+\frac{6Q_2}{\left(\frac{r}{R}\right)^4}+\frac{2S_2}{\left(\frac{r}{R}\right)^2}\right)^2\right\}\right]$$

$$=\pi\sigma^2\left[2+\frac{16S_2^2}{\left(\frac{r}{R}\right)^4}-2(1+\nu)\left\{\left(-2R_0^2-4S_2^2\right)\frac{1}{\left(\frac{r}{R}\right)^4}+\left(-24S_2Q_2-24S_2Q_2\right)\frac{1}{\left(\frac{r}{R}\right)^6}\right.\right.$$

$$\left.\left.+\left(-72Q_2^2\right)\frac{1}{\left(\frac{r}{R}\right)^8}\right\}\right]$$

$\mathcal{E}_1$ - Strain energy outside the circular region - the same at uniform stress

$$=\frac{t\sigma^2}{2E}\pi\int_a^\infty r\,dr\left[\frac{16S_2^2}{\left(\frac{r}{R}\right)^4}+2(1+\nu)\left\{\frac{2(R_0^2+2S_2^2)}{\left(\frac{r}{R}\right)^4}+\frac{48S_2Q_2}{\left(\frac{r}{R}\right)^6}+\frac{72Q_2^2}{\left(\frac{r}{R}\right)^8}\right\}\right]$$

$$=\frac{t\sigma^2}{2E}\pi R^2\left[8\frac{S_2^2}{\left(\frac{a}{R}\right)^2}+2(1+\nu)\left\{\frac{(R_0^2+2S_2^2)}{\left(\frac{a}{R}\right)^2}+12\frac{S_2Q_2}{\left(\frac{a}{R}\right)^4}+12\frac{Q_2^2}{\left(\frac{a}{R}\right)^6}\right\}\right]$$

For the extensional strain energy in the circular region

$$\widehat{rr}+\widehat{\theta\theta} = \frac{Ef}{64}\left[\left\{Q_0 + 16\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + 8\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{16}{3}f\left(\frac{r}{R}\right)^6\right\}\right.$$

$$\left. + \cos 2\theta\left\{-P_2 + 12R_2\left(\frac{r}{R}\right)^2 - \frac{16}{3}\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\int_0^{2\pi} d\theta\left\{\left(\widehat{rr}+\widehat{\theta\theta}\right)^2 - 2(1+\nu)\left(\widehat{rr}\,\widehat{\theta\theta} - \widehat{r\theta}^2\right)\right\}$$

$$= \pi\frac{E^2f^2}{64^2}\left[2\left\{Q_0 + 16\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + 8\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{16}{3}f\left(\frac{r}{R}\right)^6\right\}^2\right.$$

$$+ \left\{-P_2 + 12R_2\left(\frac{r}{R}\right)^2 - \frac{16}{3}\left(\frac{r}{R}\right)^4\right\}^2$$

$$-2(1+\nu)\left\{2\left[\frac{1}{2}Q_0 + 4\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\frac{r^2}{R^2} + \frac{4}{3}\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6\right]\right.$$

$$\times\left[\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\frac{r^2}{R^2} + \frac{20}{3}\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{14}{3}f\left(\frac{r}{R}\right)^6\right]$$

$$\left.\left. - \left[2P_2 + \frac{1}{3}\left(\frac{r}{R}\right)^4\right]\left[P_2 + 12R_2\left(\frac{r}{R}\right)^2 - 5\left(\frac{r}{R}\right)^4\right] - 4\left[P_2 + 3R_2\left(\frac{r}{R}\right)^2 - \frac{5}{6}\left(\frac{r}{R}\right)^4\right]^2\right\}\right]$$

Extensional strain energy in the circular region, E_2

$$= \frac{\pi R^2 t E f^2}{2\times 64^2}\Bigg[2\Big\{\frac{1}{2}Q_0^2\left(\frac{a}{R}\right)^2 + 8Q_0\left(\frac{a}{R}\right)^6\left(1-f\frac{a^2}{R^2}\right) + \frac{16^2}{6}\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)^2 + \frac{8}{3}Q_0\left(\frac{a}{R}\right)^6\left(2f\frac{a^2}{R^2}-1\right) - \frac{4}{3}Q_0 f\left(\frac{a}{R}\right)^8$$

$$(2f\frac{a^2}{R^2}-1)) \qquad + \frac{64}{10}\left(2f\frac{a^2}{R^2}-1\right)^2\left(\frac{a}{R}\right)^{10} +$$

$$+ 32\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right) - \frac{64}{9}f\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^{12} + \frac{8\times 16}{21}f^2\left(\frac{a}{R}\right)^{14} - \frac{32\times 1.6}{3}f\left(1-f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^{12}\Big\}$$

$$+ \Big\{\frac{1}{2}P_2^2\left(\frac{a}{R}\right)^2 - 6R_2P_2\left(\frac{a}{R}\right)^4 + \frac{16}{9}P_2\left(\frac{a}{R}\right)^6 + 24R_2^2\left(\frac{a}{R}\right)^6 - 16R_2\left(\frac{a}{R}\right)^8 + \frac{16\times 1.6}{3}\left(\frac{a}{R}\right)^{10}\Big\}$$

$$-4(1+\nu)\Big\{\frac{1}{8}Q_0^2\left(\frac{a}{R}\right)^2 + 2Q_0\left(\frac{a}{R}\right)^6\left(1-f\frac{a^2}{R^2}\right) + 8\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)^2 + \frac{2}{3}Q_0\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^6 - \frac{1}{3}Q_0 f\left(\frac{a}{R}\right)^8$$

$$+ \frac{16}{3}\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)\left(2f\frac{a^2}{R^2}-1\right) - \frac{8}{3}f\left(1-f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^{12} + \frac{8}{9}\left(2f\frac{a^2}{R^2}-1\right)^2\left(\frac{a}{R}\right)^{10} - \frac{8}{3}f\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^{12}$$

$$+ \frac{2}{7}f^2\left(\frac{a}{R}\right)^{14}\Big\}$$

$$+ 2(1+\nu)\Big\{3P_2^2\left(\frac{a}{R}\right)^2 + 12P_2R_2\left(\frac{a}{R}\right)^4 - \frac{41}{18}P_2\left(\frac{a}{R}\right)^6 - \frac{3}{4}R_2\left(\frac{a}{R}\right)^8 + \frac{1}{48}\left(\frac{a}{R}\right)^{10} + 9R_2^2\left(\frac{a}{R}\right)^6\Big\}$$

$$+ 8(1+\nu)\Big\{\frac{1}{2}P_2^2\left(\frac{a}{R}\right)^2 + \frac{3}{2}P_2R_2\left(\frac{a}{R}\right)^4 + \frac{9}{4}R_2^2\left(\frac{a}{R}\right)^6 - \frac{5}{18}P_2\left(\frac{a}{R}\right)^6 - \frac{5}{16}R_2\left(\frac{a}{R}\right)^8 + \frac{25}{360}\left(\frac{a}{R}\right)^{10}\Big\}\Bigg]$$

for the circular region

$$k_1 = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2} = \frac{1}{R}\, 4f \left\{ \left(\frac{a}{R}\right)^2 - 3\left(\frac{\rho}{R}\right)^2 \right\}$$

$$k_2 = \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2 w_0}{\partial \theta^2} - \frac{1}{\rho}\frac{\partial w_0}{\partial \rho}$$

$$= \frac{1}{R}\, 4f \left\{ \left(\frac{a}{R}\right)^2 - \left(\frac{\rho}{R}\right)^2 \right\}$$

$$\tau = 0$$

$$k_1 + k_2 = \frac{1}{R}\, 8f \left\{ \left(\frac{a}{R}\right)^2 - 2\left(\frac{\rho}{R}\right)^2 \right\}$$

$$k_1 k_2 = \frac{1}{R^2}\, 16 f^2 \left\{ \left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{\rho}{R}\right)^2 + 3\left(\frac{\rho}{R}\right)^4 \right\}$$

The bending strain energy in the circular region

$$16 f^2 \frac{1}{24}\frac{t^3}{(1-\nu^2)} E \cdot 2\pi \int_0^{\left(\frac{a}{R}\right)} \left[4\left\{ \left(\frac{a}{R}\right)^2 - 2\left(\frac{\rho}{R}\right)^2 \right\}^2 - 2(1+\nu)\left\{ \left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{\rho}{R}\right)^2 + 3\left(\frac{\rho}{R}\right)^4 \right\} \right] \frac{\rho}{R}\, d\frac{\rho}{R}$$

$$= \frac{8}{3}\frac{t^3 E \pi}{(1-\nu^2)} f^2 \int_0^{\left(\frac{a}{R}\right)} \left[2\left(\frac{a}{R}\right)^4 - 8\left(\frac{a}{R}\right)^2\left(\frac{\rho}{R}\right)^2 + 8\left(\frac{\rho}{R}\right)^4 - (1+\nu)\left\{ \left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{\rho}{R}\right)^2 + 3\left(\frac{\rho}{R}\right)^4 \right\} \right] \frac{\rho}{R}\, d\frac{\rho}{R}$$

$$= \frac{8}{3}\frac{t^3 E \pi f^2}{(1-\nu^2)} \left[1 - 2 + \frac{4}{3} - (1+\nu)\left(\frac{1}{2} - 1 + \frac{1}{2}\right) \right] \left(\frac{a}{R}\right)^6$$

$$= \frac{8}{9}\frac{t^3 E \pi f^2}{(1-\nu^2)} \left(\frac{a}{R}\right)^6 = \mathcal{E}_3 \quad \times 16 \quad [\text{due to } \underline{\underline{4f = f}}]\ !!!$$

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$$= tR^2\frac{\sigma^2}{E}\int_0^{2\pi} d\theta\Big[\tfrac{1}{2}(1-\nu)R_0 - \tfrac{1}{2}(1+\nu)R_0 + \cos^2\theta\left(-2(1+\nu)S_2 + 2S_2\right)\Big]$$

$$- d\theta\Big[-(1+\nu)S_2\Big]\sin^2\theta\Big]$$

$$= t\frac{\sigma^2}{E}\pi R^2\Big[-2\nu R_0 - 2\nu S_2 + (1+\nu)S_2\Big]$$

$$= t\frac{\sigma^2}{E}\pi R^2\Big[(1-\nu)S_2 - 2\nu R_0\Big] = f(\sigma)$$

In order to simplify the calculation: [Note: diff. from p. 309] !!!

Put $\dfrac{R_0}{\left(\frac{a}{R}\right)^2} = r_0,\qquad f\dfrac{a^2}{R^2} = g,\qquad \dfrac{Q_0}{\left(\frac{a}{R}\right)^4} = q_0$

$\dfrac{Q_2}{\left(\frac{a}{R}\right)^4} = q_2,\qquad \dfrac{S_2}{\left(\frac{a}{R}\right)^2} = s_2,\qquad \dfrac{P_2}{\left(\frac{a}{R}\right)^4} = p_2,$

$\dfrac{R_2}{\left(\frac{a}{R}\right)^2} = r_2$

Important !!! Change eqn. (A), (B), (C), (D)

With this set of notation, !!!

$$\frac{\mathcal{E}_1}{R^3} = \frac{t}{R}\frac{c^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\gamma s_2^2 + 2(1+\nu)\left\{(r_0^2 + 2 s_2^2) + [illegible]\right\}\right]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \frac{8}{9} E\pi g^2 \left(\frac{a}{R}\right)^2 \frac{1}{(1-\nu^2)} \quad \frac{1}{16}$$!!! [illegible]

$$\frac{\mathcal{Q}}{R^3} = \left(\frac{t}{R}\right)\frac{c^2}{E}\pi\left(\frac{a}{R}\right)^2\left\{(1-\nu)s_2 - 2\nu r_0\right\}$$ $\frac{r}{s}$!!!

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{1}{8192} E g^2 \left(\frac{a}{R}\right)^5 \Bigg[\Big\{ g_0^2 + 16 g_0(1-g) + \frac{256}{3}(1-g)^2 + \frac{16}{3} g_0(2g-1) - \frac{8}{3} g_0 g$$
$$+ 64(1-g)(2g-1) - \frac{128}{9} g(2g-1) + \frac{64}{9}(2g-1)^2$$
$$+ \frac{256}{21} g^2 - \frac{102.4}{3} g(1-g)\Big\}$$

$$+ \left\{\frac{1}{2}p_2^2 - 6 r_2 p_2 + \frac{11}{9} p_2 + 24 r_2^2 - 16 r_2 + \frac{256}{3}\right\}$$

$$-(1+\nu)\Big\{\frac{1}{2}g_0^2 + 8 g_0(1-g) + 32(1-g)^2 + \frac{4}{3}g_0(2g-1) - \frac{4}{3}g_0 g$$
$$+ \frac{64}{3}(1-g)(2g-1) - \frac{32}{3}g(1-g) + \frac{32}{9}(2g-1)^2 - \frac{32}{3}g(2g-1) + \frac{8}{7}g^2\Big\}$$

$$+(1+\nu)\left\{6 p_2^2 + 24 p_2 r_2 - \frac{49}{9} p_2 - \frac{3}{2} r_2 + \frac{7}{24} + 18 r_2^2\right\}\Bigg]$$

The equations for determining the values of r_0 + q_0

$$\frac{1}{2} + r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{1}{2} q_0 + 4(1-g) + \frac{2}{3}(2g-1) - \frac{2}{3} g \right\}$$

$$\frac{1}{2} - r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{1}{2} q_0 + 12(1-g) + \frac{22}{3}(2g-1) - \frac{14}{3} g \right\}$$

$$1 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ q_0 + 16(1-g) + 8(2g-1) - \frac{16}{3} g \right\}$$

$$q_0 = \frac{1}{g} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 16(1-g) - 8(2g-1) + \frac{16}{3} g$$

$$\boxed{q_0 = \frac{1}{g} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 8 + \frac{16}{3} g}$$

if $\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 = \xi$

$$\boxed{q_0 = \frac{1}{g\xi} - 8 + \frac{16}{3} g}$$

$$r_0 = \left[\frac{E}{64\sigma} \frac{a^2}{R^2}\right] g \left\{ \frac{1}{2g} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 4 + \frac{8}{3} g + \frac{8}{3} - 2g \right\} - \frac{1}{2}$$

$$\boxed{r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] g \left\{ \frac{2}{3} g - \frac{4}{3} \right\}}$$

$$\boxed{r_0 = \xi g \left(\frac{2}{3} g - \frac{4}{3} \right)}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right) \frac{E\left(\frac{?a}{?}\right)}{8(1-\nu^2)} \Big[\left\{ q_0^2 + \frac{8}{3}(4-3g) q_0 + 54.1333 - 49.7778 g + 26.4127 g^2 \right\}$$

$$+ \left\{ \frac{1}{3} p_2^2 - 6 r_2 p_2 + \frac{16}{9} p_2 + 27 r_2^2 - 15 r_2 + \frac{?}{3} \right\}$$

$$- (1+\nu) \left\{ \frac{1}{2} q_0^2 + 4\left(\frac{4}{3} - g\right) q_0 + 14.2222 - 14.2222 g - 6.2222 g^2 \right\}$$

$$+ (1+\nu) \left\{ 6 p_2^2 + 24 p_2 r_2 - \frac{49}{9} p_2 - \frac{3}{2} r_2 + 18 r_2^2 + \frac{7}{24} \right\} \Big]$$

The four equations for the unknowns r_2, s_2, q_2, p_2 are now:

$\nu = 0.3 \qquad 1+\nu = 1.3 \qquad (1+\nu)^2 = 1.69$

$1-\nu^2 = 0.91$

$$114.76\, q_2 + 46.4\, s_2 - 5.09(\xi g p_2) - 50.04(\xi g r_2) + [30.22\, \xi g - 3] = 0$$

$$46.4\, q_2 + 36\, s_2 + 5.20(\xi g p_2) + 16.8(\xi g r_2) + [13.3333\, \xi g + 1.6] = 0$$

$$-5.09\, q_2 + 5.2\, s_2 + 19.5625(\xi g p_2) + 39.63(\xi g r_2) + [2.29333\, \xi g + 5.2625] = 0$$

$$-25.02\, q_2 + 8.4\, s_2 + 19.815(\xi g p_2) + 122.50(\xi g r_2) + [-32.96\, \xi g + 7.035] = 0$$

or 1) $q_2 + 0.40432\, s_2 - 0.044353(\xi g p_2) - 0.43604(\xi g r_2) + [0.26333\, \xi g - 0.026142] = 0$

2) $q_2 + 0.77586\, s_2 + 0.11207(\xi g p_2) + 0.36207(\xi g r_2) + [0.28736\, \xi g + 0.034483] = 0$

2) $-q_2 + 1.02161\, s_2 + 3.84332(\xi g p_2) + 7.78585(\xi g r_2) + [0.45056\, \xi g + 1.03389] = 0$

1) $-q_2 + 0.33573\, s_2 + 0.79197(\xi g p_2) + 4.89608(\xi g r_2) + [-1.31735\, \xi g + 0.28118] = 0$

$$1.42593\,S_2 + 3.79197(\xi_g^0 p_2) + 7.34981(\xi_g^0 r_2) + [0.71389\,\xi_g^0 + 1.00775] = 0$$

$$1.79747\,S_2 + 3.95539(\xi_g^0 p_2) + 8.14792(\xi_g^0 r_2) + [0.73792\,\xi_g^0 + 1.06837] = 0$$

$$1.11159\,S_2 + 0.90434(\xi_g^0 p_2) + 5.25815(\xi_g^0 r_2) + [-1.02999\,\xi_g + 0.31566] = 0$$

$$S_2 + 2.66421(\xi_g^0 p_2) + 5.15440(\xi_g^0 r_2) + [0.50065\,\xi_g^0 + 0.70673] = 0$$

$$S_2 + 2.20053(\xi_g^0 p_2) + 4.53299(\xi_g^0 r_2) + [0.41053\,\xi_g^0 + 0.59437] = 0$$

$$S_2 + 0.81329(\xi_g^0 p_2) + 4.73030(\xi_g^0 r_2) + [-0.92659\,\xi_g^0 + 0.28397] = 0$$

$$1.85092(\xi_g^0 p_2) + 0.42410(\xi_g^0 r_2) + [1.42724\,\xi_g + 0.42276] = 0$$

$$0.46368(\xi_g^0 p_2) + 0.62141(\xi_g^0 r_2) + [0.09012\,\xi_g^0 + 0.11236] = 0$$

$$\xi_g^0 p_2 + 0.22913(\xi_g^0 r_2) + [0.77110\,\xi_g + 0.22841] = 0$$

$$\xi_g^0 p_2 + 1.34017(\xi_g^0 r_2) + [0.19436\,\xi_g + 0.24232] = 0$$

Thus $$\xi_g r_2 = \frac{0.57674\,\xi_g - 0.01391}{1.11104}$$

$$\boxed{\xi_g^0 r_2 = 0.51910\,\xi_g - 0.01252}$$

$$(\xi_g p_2) + 1.51930(0.51910\,\xi_g - 0.01252) + (0.91546\,\xi_g + 0.47073) = 0$$

$$\boxed{\xi_g^0 p_2 = -(0.87004\,\xi_g^0 + 0.22554)}$$

$$3S_2 - 5.67803(0.89004\,\xi^g_\ell + 0.22554) + 14.41269(0.51910\,\xi^g_\ell - 0.01252)$$
$$+ [-0.01541\,\xi^g_\ell + 1.58507] = 0$$

$$3S_2 - (5.05367\,\xi^g_\ell + 1.28062) + (7.48422\,\xi^g_\ell - 0.18051)$$
$$+ (-0.01541\,\xi^g_\ell + 1.58534) = 0$$

$$\boxed{S_2 = -(0.80505\,\xi^g_\ell + 0.04131)}$$

$$gq_2 - 1.18018(0.80505\,\xi^g_\ell + 0.04131) - 0.06772(0.89004\,\xi^g_\ell + 0.22554)$$
$$- 0.07397(0.51910\,\xi^g_\ell - 0.01252) + (0.55069\,\xi^g_\ell + 0.00834) = 0$$

$$gq_2 - (0.95010\,\xi^g_\ell + 0.04875) - (0.06027\,\xi^g_\ell + 0.01527)$$
$$- (0.03840\,\xi^g_\ell - 0.00926) + (0.55069\,\xi^g_\ell + 0.00834) = 0$$

$$\boxed{q_2 = 0.24904\,\xi^g_\ell + 0.02758}$$

We have thus

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(0.80505\,\xi g + 0.04131)^2 + 2(1+\nu)\Big\{(\xi g)^2(g-2)^2\frac{4}{9} + 2(0.80505\,\xi g + 0.04131)^2$$

$$-12\,(0.80505\,\xi g + 0.04131)(0.24904\,\xi g + 0.02736) + 12\,(0.24904\,\xi g + 0.02736)^2\Big\}\Big]$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[\,6.2\,(0.80505\,\xi g + 0.04131)^2 + 1.15556\,(\xi g)^2(g^2 - 4g + 4)$$

$$-\;31.20\,(0.24904\,\xi g + 0.02736)(0.55691\,\xi g + 0.01393)\Big]$$

$$\boxed{\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(\xi g)^2(1.15556\,g^2 - 4.62222\,g + 4.31326) - 0.17159(\xi g) - 0.001320\Big]}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a^2}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(\xi g)^2(34.5016\,g^2 - 31.2890\,g + 24.5569)$$

$$+(3.73333 - 2.8g)(\xi g)\Big\{1 - 8\left(1 - \tfrac{2}{3}g\right)(\xi g)\Big\} + 5.3(\xi g)(0.89004\,\xi g + 0.22554)$$

$$-\;17.95(\xi g)(0.5190\,\xi g - 0.01252) + 8.3\,(0.89004\,\xi g + 0.22554)^2$$

$$-\;25.20\,(0.89004\,\xi g + 0.22554)(0.5190\,\xi g - 0.01252) + 47.4(0.5190\,\xi g - 0.01252)^2\Big]$$

$$\boxed{\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\pi\frac{a^2}{R^2}\frac{\sigma^2}{2E}\Big[(\xi g)^2(19.56813\,g^2 + 11.0221\,g - 2.2066) + \xi g(5.20071 - 2.8g) + 0.50079\Big]}$$

$\frac{32}{3}$ 10.66667 6.90000 $\frac{8}{5.3}$ 4.1

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE}\left[\frac{4096}{9(1-\nu^2)}\;\frac{1}{\left(\frac{a}{R}\right)^4}(\xi g)^2\right]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE}\left[\frac{1}{9(1-\nu^2)}\;\frac{g^2}{\left(\frac{\sigma}{E}\right)^2}\right]$$

$$\boxed{\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)\pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE}\left[0.122100\;\frac{g^2}{\left(\frac{\sigma}{E\frac{t}{R}}\right)^2}\right]}$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right)\pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE}\left[-1.4\,(0.80505\,\xi g + 0.04131) - 1.2\,\xi g\left(\tfrac{2}{3}g - \tfrac{1}{3}\right)\right]$$

$$\boxed{-\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right)\pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE}\left[(\xi g)(0.8g - 0.47293) + 0.057834\right]}$$

Total potential of the system:

$$\left(\frac{t}{R}\right)\pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE}\Big[(\xi g)^2(20.7239\,g^2 + 6.3999\,g + 2.1067) + \xi g\,(5.0291 - 2.8\,g) + 0.49947$$
$$+ 0.122100\,\frac{g^2}{k^2} + \xi g\,(0.8g - 0.47293) + 0.057834\Big]$$

If σ is a compression, write

$$\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{T^2}{2E}\Big[\xi^2\left(20.7239\,g^4+6.3999\,g^3+2.1067\,g^2\right)+$$

$$-\ \xi\left(4.5562\,g-2.0\,g^2\right)+0.122100\,\frac{g^2}{k^2}+0.55730\Big]$$

Differentiate against g,

$$\xi^2\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)-\xi\left(4.5562-4.0\,g\right)$$

$$+\ 0.244200\,\frac{g}{k^2}=0$$

or $$K^2=\frac{0.244200\,g}{\xi(4.5562-4.0g)-\xi^2(82.8956\,g^3+19.1999\,g^2+4.2134\,g)}$$

or $$(4.5562-4g)=2\xi\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)$$

$$\therefore\quad \xi=\frac{1}{2}\,\frac{(4.5562-4g)}{(82.8956\,g^3+19.1999\,g^2+4.2134\,g)}$$

$$K^2=4\,\frac{0.244200\,g^2\left(82.8956\,g^2+19.1999\,g+4.2134\right)}{(4.5662-4g)^2}$$

Now if we minus the energy expression with the quantity $\left(\frac{1}{K}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{2E}\cdot 1$, so that the expression truly represents the difference of total potential of the system in two modes, then

$$\left(\frac{1}{K}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{2E}\Bigg[\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{20.7239 f^4\left(\frac{a}{R}\right)^8 + 6.3999 f^3\left(\frac{a}{R}\right)^6 + 2.1067 f^2\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562 f\left(\frac{a}{R}\right)^2 - 2.0 f^2\left(\frac{a}{R}\right)^4\right\} + 0.122100 f^2\left(\frac{a}{R}\right)^4\frac{1}{K^2} - 0.44270\Bigg]$$

the minimum condition becomes

$$\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{82.8956 f^3\left(\frac{a}{R}\right)^6 + 19.1999 f^2\left(\frac{a}{R}\right)^6 + 4.2134 f\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562\left(\frac{a}{R}\right)^2 - 4f\left(\frac{a}{R}\right)^4\right\} + 0.244200 f\left(\frac{a}{R}\right)^4\frac{1}{K^2} = 0$$

or if $\left(\frac{a}{R}\right)\neq 0$

$$\boxed{\begin{aligned}&\left(\frac{E}{64\sigma}\right)^2\left\{82.8956 f^3\left(\frac{a}{R}\right)^6 + 19.1999 f^2\left(\frac{a}{R}\right)^4 + 4.2134 f\left(\frac{a}{R}\right)^2\right\}\left(\frac{a}{R}\right)^4\\&-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562 - 4f\left(\frac{a}{R}\right)^2\right\} + f\left(\frac{a}{R}\right)^2\frac{0.244200}{K^2} = 0\end{aligned}}$$

$$\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{145.0673\, f^4\left(\frac{a}{R}\right)^8 + 38.3974\, f^3\left(\frac{a}{R}\right)^6 + 10.5335\, f^2\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{13.6686\, f\left(\frac{a}{R}\right)^2 - 8.0\, f^2\left(\frac{a}{R}\right)^4\right\} + 0.366300\, f^2\left(\frac{a}{R}\right)^4\frac{1}{K^2} - 0.44270 = 0$$

Due to very nature of the conditions, it is easier to proceed as follows.

$$\left(\frac{a}{R}\right)^4\left\{82.8956\, g^3 + 19.1999\, g^2 + 4.2134\, g\right\}\left(\frac{1}{64\, K\left(\frac{t}{R}\right)}\right)^2$$

$$-\left(\frac{a}{R}\right)^2\left\{4.5562 - 4g\right\}\frac{1}{64K\left(\frac{t}{R}\right)} + \frac{0.244200\, g}{K^2} = 0$$

or

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{64\, K\left(\frac{t}{R}\right)\left\{4.5562 - 4g\right\}}{\left\{82.8956\, g^2 + 19.1999\, g + 4.2134\right\} g}\left(\frac{a}{R}\right)^2 + \frac{0.244200 \times 64^2 \times \left(\frac{t}{R}\right)^2}{\left\{82.8956\, g^2 + 19.1999\, g + 4.2134\right\}} = 0}$$

$$\left(\frac{a}{R}\right)^4\left\{145.0673\, g^4 + 38.3974\, g^3 + 10.5335\, g^2\right\}\left(\frac{1}{64K\left(\frac{t}{R}\right)}\right)^2$$

$$-\left(\frac{a}{R}\right)^2\left\{13.6686\, g - 8g^2\right\}\frac{1}{64K\left(\frac{t}{R}\right)} + \frac{0.366300\, g^2}{K^2} - 0.44270 = 0$$

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{64K\left(\frac{t}{R}\right)\left\{13.6686 - 8g\right\}\left(\frac{a}{R}\right)^2}{\left\{145.0673\, g^2 + 38.3974\, g + 10.5335\right\} g} + \frac{0.36630 \times 64^2 \times \left(\frac{t}{R}\right)^2}{\left\{145.0673\, g^2 + 38.3974\, g + 10.5335\right\}} - \frac{0.44270 \times 64^2 \times K^2\left(\frac{t}{R}\right)^2}{\left\{145.0673\, g^2 + 38.3974\, g + 10.5335\right\} g^2} = 0}$$

Both equations can be written as

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{K\left(\frac{t}{R}\right)\left\{1.13905 - g\right\}}{\underbrace{\left\{0.32381 g^2 + 0.075000 g + 0.016459\right\}}_{X} g}\left(\frac{a}{R}\right)^2 + \frac{3.9072\left(\frac{t}{R}\right)^2}{\left\{0.32381 g^2 + 0.075000 g + 0.016459\right\}} = 0}$$

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{K\left(\frac{t}{R}\right)\left\{1.70858 - g\right\}}{\underbrace{\left\{0.28333 g^2 + 0.075000 g + 0.020573\right\}}_{W} g}\left(\frac{a}{R}\right)^2 + \frac{\left(2.9304 g^2 - 3.5416 K^2\right)\left(\frac{t}{R}\right)^2}{\left\{0.28333 g^2 + 0.075000 g + 0.020573\right\} g^2} = 0}$$

$$\left(\frac{a}{R}\right)^2 = \frac{1}{2}\,\frac{K\left(\frac{t}{R}\right)\left\{1.13905 - g\right\}}{X g} \pm \sqrt{\frac{1}{4}\,\frac{K^2\left(\frac{t}{R}\right)^2\left\{1.13905 - g\right\}^2}{X^2 g^2} - \frac{3.9072\left(\frac{t}{R}\right)^2}{X}}$$

$$= \frac{1}{2}\,\frac{K\left(\frac{t}{R}\right)\left\{1.13905 - g\right\}}{X g}\left[1 \pm \sqrt{1 - \frac{4 \times 3.9072\, X g^2}{K^2\left\{1.13905 - g\right\}^2}}\right]$$

$$\frac{K^2\left(\frac{t}{R}\right)^2}{g^2}\left[\frac{(1.70858 - g)}{W} - \frac{(1.13905 - g)}{X}\right]\frac{1}{2}\,\frac{(1.13905 - g)}{X}\left[1 \pm \sqrt{1 - \frac{4 \times 3.9072\, X g^2}{K^2 (1.13905 - g)^2}}\right]$$

$$+ \frac{\left(\frac{t}{R}\right)^2}{g^2}\left[\frac{3.9072 g^2}{X} - \frac{\left(2.9304 g^2 - 3.5416 K^2\right)}{W}\right] = 0$$

$$\frac{1}{2}\frac{(1.13905-g)}{X}\left[\frac{(1.70858-g)}{W}-\frac{(1.13905-g)}{X}\right]\left[1\pm\sqrt{1-\frac{15.6288\,X g^2}{K^2(1.13905-g)^2}}\right]K^2$$

$$+\left[\frac{3.9072g^2}{X}-\frac{(2.9304g^2-3.5416K^2)}{W}\right]=0$$

where $X = 0.32381g^2 + 0.075000g + 0.016459$

$W = 0.28333g^2 + 0.075000g + 0.020573$

When $g = 0.1$

$X = 0.027197,\quad W = 0.030906$; $\frac{1}{X} = 36.769$, $\frac{1}{W} = 32.356$

$$\frac{1}{2}\frac{1.03905}{0.027197}\left[\frac{1.60858}{0.030906}-\frac{1.03905}{0.027197}\right]\left[1\pm\sqrt{1-\frac{0.156288\times 0.027197}{K^2\times 1.03905^2}}\right]K^2$$

$$=\frac{0.029304-3.5416K^2}{0.030906}-\frac{0.039072}{0.027197}$$

$$1-\frac{0.156288\times 0.027197}{K^2\times 1.03905^2}=\left[\frac{32.356}{19.1024\times 13.8424}\left(\frac{0.029304}{K^2}-3.5416\right)-\frac{36.769\times 0.039072}{19.1024\times 13.8424}\frac{1}{K^2}-1\right]^2$$

$$1-\frac{0.0039371}{K^2}=\left[0.12237\left(\frac{0.029304}{K^2}-3.5416\right)-\frac{0.0054331}{K^2}-1\right]^2$$

$$=\left[\frac{0.0018472}{K^2}+0.56111\right]^2$$

$$1-\frac{0.0039371}{K^2}=\left(\frac{0.0018479}{K^2}\right)^2+\frac{0.0020933}{K^2}+0.32105$$

$$\left(\frac{0.001}{K^2}\right)^2 3.41215+\left(\frac{0.001}{K^2}\right)6.0304-0.67895=0$$

$$\left(\frac{0.001}{K^2}\right)^2+1.7673\left(\frac{0.001}{K^2}\right)-0.19898=0$$

$$\left(\frac{0.001}{K^2}\right)=-0.88365+\sqrt{0.88365^2+0.19898}$$

$$=-0.88365+\sqrt{0.97982}=-0.88365+0.98986$$

$$=0.10621$$

$$\therefore\quad K^2=0.0094153\qquad\boxed{k=0.097032}$$

$$\frac{\left(\frac{a}{R}\right)^2}{\left(\frac{t}{R}\right)}=\frac{1}{2}\,\frac{0.97032\times1.03905}{0.027197}\left[1\pm\sqrt{1-\frac{4\times3.9072\times0.027197\times1.0621}{1.03905^2}}\right]$$

$$=18.3845\times0.97032\times1.03905\left[1\mp\sqrt{0.58183}\right]$$

0.23722
1.76273

$$=4.397$$
32.674

If $\left(\frac{t}{R}\right)=\frac{1}{1000}$ $\quad\left(\frac{a}{R}\right)^2=0.004397\qquad\boxed{\frac{a}{R}=0.01135}$

$$\cancel{f\left(\frac{a}{R}\right)^2=0.1,\qquad\boxed{f=\frac{0.1}{0.032674}=3.060}}$$

$$\frac{w_{max}}{t}=f\,\frac{\left(\frac{a}{R}\right)^4}{4}\,\frac{R}{t}=\frac{f\left(\frac{a}{R}\right)^2/(t/R)}{4}=\frac{0.1\times\overset{32.674}{4.397}}{4}=\underset{=}{\overset{0.8169}{0.1099}}$$

If we consider (ξg) also as a variable,

$$6q_2 + 4s_2 - \frac{1}{2} = 2\xi g\, p_2 + \frac{1}{3}\xi g$$

$$6q_2 - \frac{1}{2} = \xi g\, p_2 + 12\xi g\, r_2 - 5\xi g$$

$$-6q_2 - 2s_2 - \frac{1}{2} = 2\xi g\, p_2 + 6\xi g\, r_2 - \frac{5}{3}\xi g$$

$$-0.65 - 2.6q_2 - 4s_2 = 2.3\,\xi g\, p_2 + 1.2\,\xi g\, r_2 + 4.5\,\xi g$$

$$2.6q_2 - 0.65 = 1.95\,\xi g\, p_2 + 6.6\,\xi g\, r_2 - 0.2\,\xi g$$

This is a system of equations for 5 unknowns, $q_2, s_2, p_2, r_2, (\xi g)$

$$\begin{cases} q_2 + 0.666667\, s_2 - 0.33333\,\xi g\, p_2 + 0 = 0.055556\,\xi g + 0.083333 \\ q_2 + 0 - 0.166667\,\xi g\, p_2 - 2\xi g\, r_2 = -0.833333\,\xi g + 0.083333 \\ q_2 + 0.33333\, s_2 + 0.33333\,\xi g\, p_2 + \xi g\, r_2 = +0.277778\,\xi g - 0.083333 \\ q_2 + 1.53846\, s_2 + 0.88461\,\xi g\, p_2 + 0.46154\,\xi g\, r_2 = -1.73077\,\xi g - 0.25000 \\ q_2 + 0 - 0.75000\,\xi g\, p_2 - 2.53846\,\xi g\, r_2 = -0.076923\,\xi g + 0.25000 \end{cases}$$

$$0.666667\, s_2 - 0.166667\,\xi g\, p_2 + 2\,\xi g\, r_2 = 0.888889\,\xi g$$

$$0.33333\, s_2 + 0.5000\,\xi g\, p_2 + 3\xi g\, r_2 = 1.111111\,\xi g - 0.166667$$

$$1.20513\, s_2 + 0.55128\,\xi g\, p_2 - 0.53846\,\xi g\, r_2 = -2.00855\,\xi g - 0.16667$$

$$1.53846\, s_2 + 1.63461\,\xi g\, p_2 + 3.0000\,\xi g\, r_2 = -1.65385\,\xi g - 0.50000$$

$$\delta_2 - 0.25000\,\xi_g p_2 + 3.0000\,\xi_g r_2 = 1.33333\,\xi_g$$
$$\delta_2 + 1.5000\,\xi_g p_2 + 9.000\,\xi_g r_2 = 3.33333\,\xi_g - 0.50000$$
$$\delta_2 + 0.45744\,\xi_g p_2 - 0.44681\,\xi_g r_2 = -1.66667\,\xi_g - 0.138298$$
$$\delta_2 + 1.06250\,\xi_g p_2 + 1.950\,\xi_g r_2 = -1.07500\,\xi_g - 0.32500$$

$$1.75000\,\xi_g p_2 + 6.0000\,\xi_g r_2 = 2.00000\,\xi_g - 0.50000$$
$$1.04256\,\xi_g p_2 + 9.44681\,\xi_g r_2 = 5.0000\,\xi_g - 0.36170$$
$$0.60506\,\xi_g p_2 + 2.39681\,\xi_g r_2 = 0.59167\,\xi_g - 0.18670$$

$$\xi_g p_2 + 3.42857\,\xi_g r_2 = 1.14286\,\xi_g - 0.28571$$
$$\xi_g p_2 + 9.06116\,\xi_g r_2 = 4.79589\,\xi_g - 0.34693$$
$$\xi_g p_2 + 3.96128\,\xi_g r_2 = 0.97787\,\xi_g - 0.30856$$

$$5.63259\,\xi_g r_2 = 3.65303\,\xi_g - 0.06122$$
$$5.09988\,\xi_g r_2 = 3.81802\,\xi_g - 0.03837$$

$$4.21685\,\xi_g - 0.04238 = 3.65303\,\xi_g - 0.06122$$

$$\xi_g = -\frac{0.01884}{0.56382} = -0.03341$$

If compression is taken as positive

$$\boxed{\xi_g = 0.03341}$$

$$\boxed{\xi_g = 0.03341}$$

$$r_2 = \frac{-7.47105 \times 0.03341 - 0.09959}{-10.73247 \times 0.03341}$$

$$= \frac{0.09959 + 0.24961}{0.35857} = \boxed{0.97387 = r_2}$$

$$p_2 = \frac{-0.03341(-16.02115 + 6.91662) - 0.94120}{-3 \times 0.03341}$$

$$= -3.03484 + 9.39039 = \boxed{6.35555 = p_2}$$

$$s_2 = \frac{-0.03341(-17.60449 - 13.15035 + 1.92500) - 0.96330}{4}$$

$$= 0$$

$$q_2 = \frac{-0.03341(0.20376 + 2.99652 - 2.30768) + 0.083333}{5}$$

$$= \boxed{0.01070 = q_2}$$

Check $\quad 0.01070 + 0.33333 \times 0.03341 \times 6.3555$

$= 0.08148$ $\qquad$ Checks

$$q_0 = \frac{16}{3}g - 8 + \frac{1}{g^2} = \frac{16}{3}g - 37.931$$

$$r_0 = \xi g\left(\frac{2}{3}g - \frac{4}{3}\right)$$

$$q_2 = 0.01070$$

$$s_2 = 0$$

$$p_2 = 6.3555$$

$$r_2 = 0.97347$$

$$\xi g = -0.03341$$

$$\xi g = -0.03341$$

$$= \frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 g$$

$$\left(\frac{a}{R}\right)^2 = -\frac{64 \times 0.03341}{g}\left(\frac{\sigma}{E}\right)$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{\frac{4}{9}(\xi g)^2\left(g^2 - 4g + 4\right) + 12 \times 0.01070^2\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{0.0004961\left(g^2 - 4g + 4\right) + 0.0013377\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{0.0004961\,g^2 - 0.0019844\,g + 0.0033223\right\}\right]$$

$$\boxed{\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{9E}\pi\left(\frac{a}{R}\right)^2\left[0.0012899\,g^2 - 0.0051594\,g + 0.0086380\right]}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2(0.03341)^2\Big[0.35(5.3333g - 37.931)^2 + (3.7333 - 2.8000g) \times (5.3333g - 37.931)$$

$$+ 15.6474 - 31.2889g + 34.5016g^2 + 8.9125$$

$$+ 8.3 \times 6.3555^2 + 25.2 \times 6.3555 \times 0.97367 - 5.3 \times 6.3555 + 22.05 \times 0.97367 + 7.4 \times 0.97367^2\Big]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2(0.03341)^2\Big[$$

g^2	g	
+ 9.95555	− 141.609	+ 503.536
− 14.93333	+ 106.207	− 141.609
+ 34.5016	+ 19.911	+ 15.674
29.1238	− 31.289	+ 8.913
	− 46.780	+ 335.552
		+ 155.974
		− 33.684
		+ 7.018
		+ 21.474
		872.553

$$\boxed{\begin{aligned} \frac{\mathcal{E}_2}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[0.032508g^2 - 0.052216g + 0.97394\Big] \\ \frac{\mathcal{E}_3}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[0.12210\frac{g^2}{\left(\frac{\sigma}{E}\frac{R}{t}\right)^2}\Big] \\ \frac{\delta\mathcal{E}}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[+0.013367g - 0.026733\Big]\times 2 \end{aligned}}$$

?

$$0.033798\,g^2 - 0.084109\,g + 0.03604 + \frac{0.12210\,g^2}{K^2} = 0$$

$$g^2 - 2.4886\,g + 1.0663 = 0$$

$$g = 1.2443 \pm \sqrt{1.2443^2 - 1.0663} = 1.2443 \pm \sqrt{0.4820} = 1.2443 \pm 0.6943$$

$$= 0.5500$$

$$1.9386$$

$$\boxed{K^2 = \frac{g^2}{0.68885\,g - (0.29517 + 0.27681\,g^2)}}$$

①	②	③	④	⑤	⑥	⑦	⑧
g	g^2	$0.68885g$	$0.27681g^2$	③ − (0.29517 + ④)	③/⑤ = K^2		
0.60	0.36	0.41331	0.09965	0.01849	19.47		
0.70	0.49	0.48220	0.13564	0.05139	9.53		
0.80	0.64	0.55108	0.17716	0.07875	8.13		
0.90	0.81	0.61997	0.22422	0.10058	8.05		
1.00	1.00	0.68885	0.27681	0.11687	8.55		
1.10	1.21	0.75774	0.33494	0.12763	9.49		
1.20	1.44	0.82662	0.39861	0.13284	10.84		
1.40	1.69	0.96439	0.46781	0.20141			
1.60	2.56	1.10216	0.70863	0.09836			
1.[illegible]	3.24	1.23993	0.89686	0.04790			

Comparison of different energies

For $g = 0.90$,

(Difference!)

$\frac{\mathcal{E}_1}{R^3}$ = extensional strain energy outside the circular region

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\begin{array}{r} 0.001044 \\ -0.004640 \\ 0.008638 \\ \hline 0.005042 \end{array}\right]$$

$$\xi g = -0.03341$$

$$\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 g = -0.03341$$

$$\left(\frac{a}{R}\right)^2 = \frac{-64 \times 0.03341}{g}\frac{\sigma}{E}$$

For the circular region:

$$\widehat{rr} = \frac{Ef}{64}\left[\left\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}\left(3f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6\right\} - \cos 2\theta\left\{2P_2 + \frac{1}{3}\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\widehat{\theta\theta} = \frac{Ef}{64}\left[\left\{\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{20}{3}\left(3f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{14}{3}f\left(\frac{r}{R}\right)^6\right\} + \cos 2\theta\left\{2P_2 + 12R_2\left(\frac{r}{R}\right)^2 - 5\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\widehat{r\theta} = \frac{Ef}{64}\left[\sin 2\theta\left\{2P_2 + 6R_2\left(\frac{r}{R}\right)^2 - \frac{5}{3}\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\frac{1}{E}\left(\widehat{rr} - \nu\widehat{\theta\theta}\right) = \frac{f}{64}\left[\left\{(1-\nu)\frac{Q_0}{2} + 4(1-3\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}(1-5\nu)\left(3f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}(1-7\nu)f\left(\frac{r}{R}\right)^6\right\} - \cos 2\theta\left\{2(1+\nu)P_2 + 12\nu R_2\left(\frac{r}{R}\right)^2 + \left(\frac{1}{3} - 5\nu\right)\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{\partial w_0}{\partial r}\right)^2\right\} = \frac{f}{64}\left\{32\left(\frac{a}{R}\right)^2\left(f\frac{a^2}{R^2} - 1\right)\frac{r^2}{R^2} - 32\left(3f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 + 32f\left(\frac{r}{R}\right)^6 + \cos 2\theta\left(32\left(\frac{a}{R}\right)^2 - 32\left(\frac{r}{R}\right)^2\right)\left(\frac{r}{R}\right)^2\right\}$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{f}{64}\left[\left\{(1-\nu)\frac{Q_0}{2} + 12(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{20}{3}(5-\nu)\left(3f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{14}{3}(7-\nu)f\left(\frac{r}{R}\right)^6\right\} - \cos 2\theta\left\{2(1+\nu)P_2 + 4\left(8\frac{a^2}{R^2} + 3\nu R_2\right)\left(\frac{r}{R}\right)^2 - 5\left(\frac{19}{3} + \nu\right)\left(\frac{r}{R}\right)^4\right\}\right]$$

$$\frac{u}{R} = \frac{f}{64}\Big[\Big\{\frac{(1-\nu)}{2} Q_0 \left(\frac{R}{a}\right) + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^3 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^5$$
$$- \frac{2}{3}(7-\nu) f \left(\frac{a}{R}\right)^7\Big\} - \cos 2\theta \Big\{ 2(1+\nu) P_2 \left(\frac{R}{a}\right) + \frac{4}{3}\left(8\frac{a^2}{R^2} + 3\nu R_2\right)\left(\frac{a}{R}\right)^3$$
$$- \left(\frac{19}{3} + \nu\right)\left(\frac{a}{R}\right)^5 \Big\}\Big]$$

$$\frac{1}{E}(\widehat{rr} - \nu\,\widehat{\theta\theta}) = \frac{f}{64}\Big[\Big\{\frac{(1-\nu)}{2} Q_0 + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^2 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4$$
$$- \frac{2}{3}(7-\nu) f \left(\frac{a}{R}\right)^6\Big\} + \cos 2\theta \Big\{ 2(1+\nu) P_2 + 12\, R_2 \left(\frac{a}{R}\right)^2 - \left(5 - \frac{1}{3}\nu\right)\left(\frac{a}{R}\right)^4 \Big\}\Big]$$

$$\frac{1}{R}\frac{\partial v}{\partial \theta} = \frac{f}{64} \cos 2\theta \Big\{ 4(1+\nu) P_2 + \frac{32}{3}\left(\frac{a}{R}\right)^2\left(\frac{a}{R}\right)^2 + 12(1+\nu) R_2 \left(\frac{a}{R}\right)^2 - \frac{2}{3}(17+\nu)\left(\frac{a}{R}\right)^4 \Big\}$$

$$\boxed{\frac{v}{R} = \frac{f}{64} \sin 2\theta \Big\{ 2(1+\nu) P_2 \left(\frac{R}{a}\right) + \frac{16}{3}\left(\frac{a}{R}\right)^2\left(\frac{a}{R}\right)^3 + 6(1+\nu) R_2 \left(\frac{a}{R}\right)^3 - \frac{1}{3}(17+\nu)\left(\frac{a}{R}\right)^5 \Big\}}$$

For the region outside the circle.

$$\widehat{rr} = \sigma\Big[\frac{1}{2} + \frac{P_0}{\left(\frac{a}{R}\right)^2} + \cos 2\theta \Big\{\frac{1}{2} - \frac{6 Q_2}{\left(\frac{a}{R}\right)^4} - \frac{4 S_2}{\left(\frac{a}{R}\right)^2}\Big\}\Big]$$

$$\widehat{\theta\theta} = \sigma\Big[\frac{1}{2} - \frac{P_0}{\left(\frac{a}{R}\right)^2} + \cos 2\theta \Big\{\frac{6 Q_2}{\left(\frac{a}{R}\right)^4} - \frac{1}{2}\Big\}\Big]$$

$$\widehat{r\theta} = -\sigma \sin 2\theta \Big\{\frac{1}{2} + \frac{6 Q_2}{\left(\frac{a}{R}\right)^4} + \frac{2 S_2}{\left(\frac{a}{R}\right)^2}\Big\}$$

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$$\frac{u}{R} = \frac{\sigma}{E}\left[\frac{1}{2}(1-\nu)\left(\frac{r}{R}\right) - (1+\nu)\frac{P_0}{\left(\frac{r}{R}\right)} + \cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{r}{R}\right) + 2(1+\nu)\frac{Q_2}{\left(\frac{r}{R}\right)^3} + \frac{4S_2}{\left(\frac{r}{R}\right)}\right\}\right]$$

$$\frac{v}{R} = \frac{\sigma}{E}\sin 2\theta\left\{2(1+\nu)\frac{Q_2}{\left(\frac{r}{R}\right)^3} - \frac{1}{2}(1+\nu)\left(\frac{r}{R}\right)\right\}$$

With the simplified relation of p. 316, the condition of stress continuity becomes

$$\frac{1}{2} + r_0 = \xi g\left\{\frac{1}{2}q_0 + 4(1-g) + \frac{4}{3}(2g-1) - \frac{2}{3}g\right\}$$

$$-\frac{1}{2} + 6q_2 + 4S_2 = \xi g\left\{2p_2 + \frac{1}{3}\right\}$$

$$\frac{1}{2} - r_0 = \xi g\left\{\frac{1}{2}q_0 + 12(1-g) + \frac{20}{3}(2g-1) - \frac{14}{3}g\right\}$$

$$6q_2 - \frac{1}{2} = \xi g\left\{2p_2 + 12r_2 - 5\right\}$$

$$-\frac{1}{2} - 6q_2 - 2S_2 = \xi g\left\{2p_2 + 6r_2 - \frac{5}{3}\right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4S_2 = \xi g\left\{-2(1+\nu)p_2 - \frac{4}{3}(8+30r_2) + \left(\frac{19}{3}+\nu\right)\right\}$$

$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = \xi g\left\{2(1+\nu)p_2 + 6(1+\nu)r_2\ \frac{1}{3}(1+\nu)\right\}$$

Thus
$$\boxed{\begin{aligned} q_0 &= \frac{1}{g\xi} - 8\left(1-\frac{2}{3}g\right) \\ r_0 &= \xi g\,\frac{2}{3}(g-2). \end{aligned}}$$

$$q_2 + 0.666667\, s_2 - 0.083333 = \xi g\,\{0.33333\, p_2 + 0.055556\}$$

$$q_2 + 0 \qquad - 0.083333 = \xi g\,\{0.33333\, p_2 + 2 r_2 - 0.833333\}$$

$$-q_2 - 0.33333\, s_2 - 0.083333 = \xi g\,\{0.33333\, p_2 + r_2 - 0.277778\}$$

$$q_2 + 1.53846\, s_2 + 0.25000 = \xi g\,\{- p_2 - 0.461538\, r_2 - 1.55128\}$$

$$q_2 \qquad - 0.25000 = \xi g\,\{p_2 + 3 r_2 - 0.155128\}$$

$$0.666667\, s_2 + 0 = \xi g\,\{-2 r_2 + 0.888889\}$$

$$-0.33333\, s_2 - 0.166667 = \xi g\,\{0.66667\, p_2 + 3 r_2 - 1.111111\}$$

$$1.20513\, s_2 + 0.166667 = \xi g\,\{-0.66667\, p_2 + 0.538462\, r_2 - 1.82906\}$$

$$1.53846\, s_2 + 0.50000 = \xi g\,\{-2 p_2 - 3.461538\, r_2 - 1.39615\}$$

$$s_2 + 0 = \xi g\,\{-3 r_2 + 1.33333\}$$

$$-s_2 - 0.5000 = \xi g\,\{2 p_2 + 9 r_2 - 3.33333\}$$

$$s_2 + 0.138298 = \xi g\,\{-0.553191\, p_2 + 0.446808\, r_2 - 1.51773\}$$

$$s_2 + 0.325 = \xi g\,\{-1.3\, p_2 - 2.25000\, r_2 - 0.907498\}$$

$$2\xi g\, p_2 + 6\xi g\, r_2 = 2\xi g - 0.5000$$

$$1.446809\,\xi g\, p_2 + 9.446808\,\xi g\, r_2 = 4.85106\,\xi g - 0.361702$$

$$0.74681\,\xi g\, p_2 + 2.196808\,\xi g\, r_2 = 0.61023\,\xi g - 0.186702$$

$$\xi_g p_2 + 3\xi_g r_2 = \xi_g - 0.250000$$

$$\xi_g p_2 + 6.52941\, \xi_g r_2 = 3.35294\, \xi_g - 0.25000$$

$$\xi_g p_2 + 3.61111\, \xi_g r_2 = 0.817116\, \xi_g - 0.25000$$

$$\left.\begin{aligned} 3.52941\, \xi_g r_2 &= 2.35294\, \xi_g \\ 2.91830\, \xi_g r_2 &= 2.53562\, \xi_g \end{aligned}\right\}$$ Impossible

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Method of Least Square :

$$\boxed{q_2 + 0.507692\, s_2 - 0.0166667 = \xi_g \{ 0.0666667\, p_2 + 0.707692\, r_2 - 0.441281 \}}$$

$$1.5000\, q_2 + s_2 - 0.12500 = \xi_g \{ 0.5000\, p_2 + 0.083333 \}$$

$$3\, q_2 + s_2 + 0.25000 = \xi_g \{ - p_2 - 3 r_2 + 0.833333 \}$$

$$0.65\, q_2 + s_2 + 0.1625 = \xi_g \{ -0.65\, p_2 - 0.3000\, r_2 - 1.00833 \}$$

$$\boxed{q_2 + 0.582525\, s_2 + 0.0551253 = \xi_g \{ -0.223301\, p_2 - 0.640778\, r_2 - 0.0177794 \}}$$

$$0.666667\, q_2 + 0.444444\, s_2 - 0.055556 = \xi\frac{g}{l}\left\{0.222222\, p_2 + 0.0370371\right\}$$

$$0.333333\, q_2 + 0.1111111\, s_2 + 0.0277777 = \xi\frac{g}{l}\left\{-0.111111\, p_2 - 0.33333\, r_2 + 0.0925926\right\}$$

$$1.53846\, q_2 + 2.36686\, s_2 + 0.384615 = \xi\frac{g}{l}\left\{-1.53846\, p_2 - 0.710058\, r_2 - 2.38658\right\}$$

$$2.53846\, q_2 + 2.92241\, s_2 + 0.356836 = \xi\frac{g}{l}\left\{-1.42735\, p_2 - 1.043391\, r_2 - 2.25695\right\}$$

$$\boxed{q_2 + 1.15125\, s_2 + 0.140573 = \xi\frac{g}{l}\left\{-0.562289\, p_2 - 0.411032\, r_2 - 0.889101\right\}}$$

$$0.333333\, q_2 + 0.222222\, s_2 - 0.0277777 = \xi\frac{g}{l}\left\{0.111111\, p_2 + 0.0185185\right\}$$

$$0.33333\, q_2 \qquad - 0.0277777 = \xi\frac{g}{l}\left\{0.111111\, p_2 + 0.666667\, r_2 - 0.277777\right\}$$

$$-0.33333\, q_2 - 0.11111\, s_2 - 0.0277777 = \xi\frac{g}{l}\left\{0.11111\, p_2 + 0.33333\, r_2 - 0.0925926\right\}$$

$$-\, q_2 - 1.53846\, s_2 - 0.25000 = \xi\frac{g}{l}\left\{p_2 + 0.461538\, r_2 + 1.55128\right\}$$

$$q_2 \qquad - 0.25000 = \xi\frac{g}{l}\left\{p_2 + 3\, r_2 - 0.155128\right\}$$

$$0.33333\, q_2 - 1.42735\, s_2 - 0.583333 = \xi\frac{g}{l}\left\{2.33333\, p_2 + 4.461538\, r_2 + 1.04430\right\}$$

$$\boxed{q_2 - 4.28205\, s_2 - 1.75000 = \xi\frac{g}{l}\left\{7\, p_2 + 13.3846\, r_2 + 3.13290\right\}}$$

$$2q_2 + 0 \quad - 0.1666667 = \xi g\{0.666667\, p_2 + 7 r_2 - 1.666667\}$$

$$-q_2 - 0.33333\, s_2 - 0.083333 = \xi g\{0.33333\, p_2 + r_2 - 0.277778\}$$

$$-0.461538\, q_2 - 0.710058\, s_2 - 0.115385 = \xi g\{0.461538\, p_2 + 0.213017\, r_2 + 0.715975\}$$

$$3 q_2 \qquad\qquad - 0.25000 = \xi g\{3 p_2 + 9 r_2 - 0.465384\}$$

$$3.538462\, q_2 - 1.043391\, s_2 - 1.115385 = \xi g\{4.461538\, p_2 + 14.213017\, r_2 - 1.693857\}$$

$$\boxed{q_2 - 0.294871\, s_2 - 0.315218 = \xi g\{1.26087\, p_2 + 4.01672\, r_2 - 0.478698\}}$$

The equations for constants are then

$$q_2 + 0.507692\, s_2 - 0.016667\, \xi g\, p_2 - 0.707692\, \xi g\, r_2 = -0.441281\, \xi g + 0.016667$$

$$q_2 + 1.15125\, s_2 + 0.562289\, \xi g\, p_2 + 0.711039\, \xi g\, r_2 = -0.889131\, \xi g - 3.140573$$

$$q_2 - 4.28205\, s_2 - 7.00000\, \xi g\, p_2 - 13.3846\, \xi g\, r_2 = 3.13290\, \xi g + 1.750000$$

$$q_2 - 0.294871\, s_2 - 1.26087\, \xi g\, p_2 - 4.01672\, \xi g\, r_2 = -0.478698\, \xi g + 0.315218$$

$$0.64356\, s_2 + 0.628956\, \xi g\, p_2 + 1.12724\, \xi g\, r_2 = -0.447820\, \xi g - 0.157240$$

$$5.43330\, s_2 + 7.562289\, \xi g\, p_2 + 13.79563\, \xi g\, r_2 = -4.02200\, \xi g - 1.890573$$

$$3.98718\, s_2 + 5.73913\, \xi g\, p_2 + 9.36788\, \xi g\, r_2 = -3.61160\, \xi g - 1.43478$$

$$s_2 + 0.977310\, \xi g\, p_2 + 1.73834\, \xi g\, r_2 = -0.695850\, \xi g - 0.244329$$

$$s_2 + 1.391839\, \xi g\, p_2 + 2.53909\, \xi g\, r_2 = -0.740249\, \xi g - 0.347960$$

$$s_2 + 1.439397\, \xi g\, p_2 + 2.34950\, \xi g\, r_2 = -0.905804\, \xi g - 0.357849$$

$$0.462087\, \xi g\, p_2 + 0.61116\, \xi g\, r_2 = -0.209954\, \xi g - 0.115520$$

$$0.414529\, \xi g\, p_2 + 0.80075\, \xi g\, r_2 = -0.044399\, \xi g - 0.103631$$

$$\xi g\, p_2 + 1.32261\, \xi g\, r_2 = -0.454359\, \xi g - 0.249996$$

$$\xi g\, p_2 + 1.93171\, \xi g\, r_2 = -0.107107\, \xi g - 0.249996$$

$$0.60910\, \xi g\, r_2 = 0.347252\, \xi g$$

$$\boxed{\xi_g \mu_2 = 0.570106\,\xi_g}$$

$$2\xi_g \lambda_2 = -2.41678\,\xi_g - 2\times 0.25000$$

$$\boxed{\xi_g \lambda_2 = -1.20839\,\xi_g - 0.25000}$$

$$3 s_2 = (4.60233 - 3.77805 - 2.34190)\,\xi_g + 0.952163 - 0.952138$$

$$\boxed{s_2 = -0.505873\,\xi_g}$$

$$4 q_2 = \xi_g(-1.47613 - 9.38345 + 10.08972 + 1.32382) - 1.94131 + 1.94131$$

$$\boxed{q_2 = 0.138490\,\xi_g}$$

Check: ξ_g

$$\begin{array}{r} +0.138490 \\ -0.256828 \\ +0.080559 \\ -0.403459 \\ \hline -0.441238 \end{array} \qquad + 0.0166667$$

O.K.

The extensional strain energy in the circular region

$$\hat{R}_1 + \hat{R}_2 = \frac{Ef}{64}\Big[\Big\{Q_0 + 16\frac{a^2}{R^2}\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{a}{R}\Big)^2 + 8\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{a}{R}\Big)^4 - \frac{16}{3}f\Big(\frac{a}{R}\Big)^6\Big\}$$

$$+ \cos 2\theta \Big\{12 R_2 \Big(\frac{a}{R}\Big)^2 - \frac{16}{3}\Big(\frac{a}{R}\Big)^4\Big\}\Big]$$

$$-\Big\{2P_2 + \frac{1}{3}\Big(\frac{a}{R}\Big)^4\Big\}\Big\{2P_2 + 12R_2\Big(\frac{a}{R}\Big)^2 - 5\Big(\frac{a}{R}\Big)^4\Big\} - \Big\{2P_2 + 6R_2\Big(\frac{a}{R}\Big)^2 - \frac{5}{3}\Big(\frac{a}{R}\Big)^4\Big\}^2$$

$$= -\Big[4P_2^2 + \frac{2}{3}P_2\Big(\frac{a}{R}\Big)^4 + 24P_2R_2\Big(\frac{a}{R}\Big)^2 + 4R_2\Big(\frac{a}{R}\Big)^6 - 10P_2\Big(\frac{a}{R}\Big)^4 - \frac{5}{3}\Big(\frac{a}{R}\Big)^6\Big]$$

$$-\Big[4P_2^2 + 36R_2^2\Big(\frac{a}{R}\Big)^4 + \frac{25}{9}\Big(\frac{a}{R}\Big)^8 + 24P_2R_2\Big(\frac{a}{R}\Big)^2 - \frac{20}{3}P_2\Big(\frac{a}{R}\Big)^4 - 20R_2\Big(\frac{a}{R}\Big)^6\Big]$$

$$= -\Big[8P_2^2 + 36R_2^2\Big(\frac{a}{R}\Big)^4 + 48P_2R_2\Big(\frac{a}{R}\Big)^2 - 16P_2\Big(\frac{a}{R}\Big)^4 - 16R_2\Big(\frac{a}{R}\Big)^6 + \frac{10}{9}\Big(\frac{a}{R}\Big)^8\Big]$$

$$+ 2(1+\nu)\Big[4P_2^2\Big(\frac{a}{R}\Big)^2 + 6R_2^2\Big(\frac{a}{R}\Big)^6 + 12P_2R_2\Big(\frac{a}{R}\Big)^4 - \frac{8}{3}P_2\Big(\frac{a}{R}\Big)^6 - 2R_2\Big(\frac{a}{R}\Big)^8 + \frac{1}{9}\Big(\frac{a}{R}\Big)^{10}\Big]$$

$$+ \Big[24R_2^2\Big(\frac{a}{R}\Big)^6 - 16R_2\Big(\frac{a}{R}\Big)^8 + \frac{16\times1.6}{3}\Big(\frac{a}{R}\Big)^{10}\Big]$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[s_2^2 + 2(1+\nu)\left\{(r_0^2+2s_2^2)+12\,s_2 q_2 + 12 q_2^2\right\}\right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \xi^2 g^2\left[\left\{q_0^2+\frac{8}{3}(4-3g)q_0 + 34.1333 - 49.7778g + 26.4127g^2\right\}\right.$$

$$-(1+\nu)\left\{\frac{1}{2}q_0^2 + 4\left(\frac{4}{3}-g\right)q_0 + 14.2222 - 14.2222g - 6.2222g^2\right\}$$

$$+\left\{24 r_2^2 - 16 r_2 + \frac{25.6}{3}\right\}$$

$$\left. + (1+\nu)\left\{8p_2^2 + 12 r_2^2 + 24 p_2 r_2 - \frac{16}{3}p_2 - 4r_2 + \frac{2}{9}\right\}\right]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \quad 0.122100\,\frac{g^2}{K^2}$$

$$\frac{p}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{2(1-\nu)s_2 - 4\nu r_0\right\}$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[0.50587^2\xi^2 g^2 + 2.6\left\{\xi^2 g^2\frac{4}{9}(g^2-4g+4) + 2\times 0.50587^2\xi^2 g^2\right.\right.$$

$$\left.\left. - 12\times 0.13849\times 0.50587\,\xi^2 g^2 + 12\times 0.13849^2\,\xi^2 g^2\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\xi^2 g^2\left(\begin{matrix}1.58660\\1.77777\\-0.84070\\+0.23015\end{matrix} \quad - 1.77777g \quad + 0.44444g^2\right)\right]$$

[illegible]

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\xi^2 g^2\left(0.44444g^2 - 1.77776g + 2.75383\right)\right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \xi^2 g^2 \Big[\Big\{0.35\,q_0^2 + (3.7333 - 2.80000\,g)\,q_0$$

$$+ 15.6445 - 31.2890\,g + 34.5015\,g^2\Big\}$$

$$+\Big\{10.4\,p_2^2 + 39.6\,r_2^2 + 31.2\,p_2 r_2 - 6.93333\,p_2 - 21.2\,r_2 + 8.82222\Big\}\Big]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\,0.35\big(1 - 8(\xi g) + 5.33333(\xi g)g\big)^2$$

$$+ \xi g\,(3.7333 - 2.80000\,g)\big(1 - 8(\xi g) + 5.3333(\xi g)g\big)$$

$$+ (15.6445 - 31.2890\,g + 34.5015\,g^2)\,\xi^2 g^2$$

$$+ 10.4\times(1.20839\,\xi g + 0.25000)^2 + 39.6\times 0.570106\,\xi^2 g^2 - 31.2\times 0.570106\,\xi g \times(1.20839\,\xi g + 0.25000)$$

$$+ 6.93333\,\xi g\,(1.20839\,\xi g + 0.25000) - 21.2\,\xi g\times 0.570106\,\xi g + 8.82222\,\xi^2 g^2\Big\}\Big]$$

$\xi^2 g^2 \Big[$

g^2 9.95555	g −29.8666	+ 22.40000
−14.93333	+ 49.3111	− 29.86666
+34.5015	−31.2890	+ 15.6445
		+ 15.1862
		+ 22.5762
		− 21.4940
		+ 8.3782
		− 12.0862
		+ 8.8222

$$(\xi g)\Big[\; g \begin{array}{l} 3.73333 \\ -2.80000 \end{array} \qquad \begin{array}{l} -\ 5.6 \\ +\ 3.73333 \\ +\ 6.28363 \\ -\ 4.44683 \\ +\ 1.73333 \end{array} \Big]$$

$$+\big[\;0.35 + 0.65\big]$$

$$\boxed{\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\,\xi^2 g^2\left(29.5237\,g^2 - 18.8445 + 29.5605\right) + \xi g\left(0.93333\,g + 1.70346\right)\Big].}$$

$$\frac{\mathcal{V}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a^2}{R^2}\right)\Big\{-1.4\times 0.50587\,\xi g - 1.2\times 0.66167\,(g - 2)\,\xi g\Big\}$$

$$\boxed{\frac{\mathcal{V}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big\{0.89178 - 0.80000\,g\Big\}(\xi g)}$$

$$\frac{\mathcal{E}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big\{\,\xi^2 g^2\left(29.9681\,g^2 - 20.6223\,g + 32.3143\right) + \xi g\left(1.73333\,g + 0.81168\right) + 0.122133\,\frac{g^2}{K^2}\Big\}$$

Let σ be compression,

$$K_\sigma^2 = \frac{0.122100\, g}{\xi(1.73333\, g + 0.81168) - \xi^2(29.9681 g^3 - 20.6223 g^2 + 32.3143 g)}$$

$$\xi = \frac{1}{2}\,\frac{1.733333 g + 0.81168}{g(29.9681 g^2 - 20.6223 g + 32.3143)} = \frac{1}{64}\frac{1}{(\frac{t}{R})}\frac{1}{K}\left(\frac{a}{R}\right)^2$$

$$K^2 = \frac{0.488400\, g^2(29.9681 g^2 - 20.6223 g + 32.3143)}{(1.7333\, g + 0.81168)^2}$$

g = amplitude factor

When $g = \frac{0.89178}{0.8000} = 1.1147$

$$K^2 = \frac{0.4884 \times 1.2426 \times (37.238 - 22.988 + 32.3143)}{(0.93333}$$

$g = 0.1$

$$K = 0.1\,\frac{\sqrt{0.488400(30.5518)}}{0.98501} = \underline{\underline{0.3920}}$$

$$\left(\frac{a}{R}\right)^2 = 32\left(\frac{t}{R}\right)\frac{\sqrt{0.488400}}{\sqrt{29.9681 g^2 - 20.6223 g + 32.3143}} = 4.04\left(\frac{t}{R}\right)$$

$\frac{t}{R} = \frac{1}{1000}$, $\frac{a}{R} = 0.0636$

At $g = 0.1$

$$\xi^2 g^2 = \frac{1}{4} \frac{(1.73333g + 0.81168)^2}{(29.9681g^2 - 20.6223g + 32.3143)^2} = 0.00025985$$

$$\xi g = \frac{1}{2} \frac{1.7333g + 0.81168}{29.9681g^2 - 20.6223g + 32.3143} = \frac{1}{2} \frac{0.98501}{30.5518} = 0.016120$$

$$\mathcal{E}_1 \sim 0.00025985 \times 2.58049 = \underline{0.00067054} \; +$$ Energy outside the circular region

$$\mathcal{E}_2 \sim 0.00025985 \times 27.9713 - 0.016120 \times 1.79679$$

$$= 0.007268 - 0.028964 = \underline{-0.021696}$$ Energy — external in the region

$$\mathcal{E}_3 \sim 0.122100 \frac{0.01}{0.3920^2} = \underline{+0.00795}$$ Bending Energy

$$\wp \sim -0.016120 \times 0.81178 = \underline{-0.013086} \; (?)$$ Increase in [illegible]

$$\frac{W_{mon}}{t} = f \frac{1}{4} \left(\frac{a}{R}\right)^4 \left(\frac{R}{t}\right) = \frac{g}{4} \left(\frac{a}{R}\right)^2 \left(\frac{R}{t}\right) = \frac{4.04}{4} g = 0.101$$ (Too small)

by virtual work

$$\frac{P}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{-2(1+\nu)\eta_0 + 4\delta_2\right\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{-\xi g \times 26 \times 0.111117(g-2) - 2.02349\,\xi g\right\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{1.44318 - 1.73333\,g\right\}\xi g$$

$$k^2 = \frac{0.4884\,g\,(29.9681\,g^2 - 20.6223\,g + 32.3143)}{(2.6666\,g + 0.26028)^2}$$

$k = 1.2$[illegible]

$g = 0.1$ $\qquad k = \frac{0.0699\sqrt{30.55}}{0.5[illegible]}$ $= 0.$[illegible]

$g = 0.05$ $\qquad k = \frac{0.03495}{0.39361}\sqrt{31.3[illegible]} = 0.498$

$g = 0.15$ $\qquad k = \frac{0.1048\sqrt{29.917}}{0.6598} = 0.870$

~~Take~~ $\cancel{2.91830\,\xi g r_2 = 2.53582\,\xi g}$

~~or~~ $\boxed{\cancel{\xi g r_2 = 0.86894\,\xi g}}$

If we drop the condition of continuity of u,

$$q_2 + 0.66667\,s_2 - 0.083333 = \xi g\{0.33333\,p_2 + 0.055556\}$$
$$q_2 \qquad - 0.083333 = \xi g\{0.3333\,p_2 + 2r_2 - 0.33333\}$$
$$-q_2 - 0.33333\,s_2 - 0.083333 = \xi g\{0.33333\,p_2 + r_2 - 0.277778\}$$
$$q_2 \qquad - 0.250000 = \xi g\{p_2 + 3r_2 - 0.155178\}$$

$$0.666667\,s_2 + 0 = \xi g\{-2r_2 + 0.[illegible]\}$$
$$-0.33333\,s_2 - 0.166667 = \xi g\{0.66667\,p_2 + 3r_2 - 1.111111\}$$
$$-0.33333\,s_2 - 0.33333 = \xi g\{1.3333\,p_2 + 4r_2 - 0.432103\}$$

$$s_2 + 0 = \xi g\{-3r_2 + 1.33333\}$$
$$-s_2 - 0.5000 = \xi g\{2p_2 + 9r_2 - 3.33333\}$$
$$-s_2 - 1.000 = \xi g\{4p_2 + 12r_2 - 1.296309\}$$

$$2\xi g p_2 + 6\xi g r_2 = 2\xi g - 0.50000$$
$$2\xi g p_2 + 3\xi g r_2 = -2.03462\,\xi g - 0.5000$$

$$3\xi g r_2 = 4.03462\,\xi g$$

$$\boxed{\xi g r_2 = 1.34487\, \xi g}$$

$$\xi g p_2 = \frac{1}{4}\left\{(-12.1038 - 0.03462)\,\xi g - 1.000\right\}$$

$$\boxed{\xi g p_2 = -3.034675\, \xi g - 0.2500}$$

$$3 s_2 + 1.5000 = \xi g\left\{-6 p_2 - 24 r_2 + 5.96537\right\}$$

$$= \xi g\left\{18.2081 - 32.27688 + 5.96537\right\} + 1.5000$$

$$\boxed{s_2 = -2.70113\, \xi g}$$

$$4 q_2 + s_2 - 0.33333 = \xi g\left\{1.3333\, p_2 + 4 r_2 - 0.655127\right\}$$

$$q_2 = \frac{1}{4}\left\{2.70113 - 4.04623 + 5.37948 - 0.65513\right\}\xi g$$

$$\boxed{q_2 = 0.84481\, \xi g}$$

$$\frac{\mathcal{E}_1}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\,2.70113^2+2.6\Big\{\frac{4}{9}(g^2-4g+4)+2\times 2.70113^2$$

$$-12\times 2.70113\times 0.84481+12\times 0.84481^2\Big\}\Big](\xi g)^2$$

$$=\left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\,0.44444\,g^2-1.77778\,g+\begin{array}{r}1.77778\\7.29610\\37.93972\\-27.38330\\+8.56445\end{array}\right](\xi g)^2$$

$$\frac{\mathcal{E}_1}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\,0.44444\,g^2-1.77778\,g+28.19475\Big](\xi g)^2$$

$$\frac{\mathcal{E}_2}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\,0.35\,(1-8\,\xi g+5.2333\,(\xi g)^2)^2+\xi g\,(3.7333-2.6\ldots)(1-8\,\xi g+5.333\,\xi g)$$

$$+(34.5015\,g^2-31.2810\,g+15.6445)\,\xi g^2$$

$$+10.4\,(3.03425\,\xi g+0.2500)^2+39.6\times 1.34467^2(\xi g)^2$$

$$-\;35.02661\,\xi g\,(3.034675\,\xi g+0.2500)-28.51124\,(\xi g)^2+8.8222\,\xi g)^2\Big]$$

$$(\xi g)^2\left[\,29.5237\,g^2-18.8445\,g+\begin{array}{r}22.40000\\-29.86666\\+15.6445\\+95.7810\\+71.6237\\-106.2944\\-28.5112\\+8.8222\end{array}\right]$$

$$\xi g \Big[\begin{array}{l} 3.73333\, g \\ -2.80000 \end{array} \qquad \begin{array}{r} -5.6 \\ 3.73333 \\ +15.7807 \\ -8.7567 \\ \hline \end{array} \Big]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[(\xi g)^2\left(29.5237 g^2 - 18.8445 g + 49.5991\right)$$

$$+ \xi g\left(0.93333 g + 5.1573\right)\Big]$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big\{-\xi g \times 2.6 \times 0.66667\,(g-2) - 10.8045\,\xi g\Big\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2(\xi g)\Big\{-1.73333\, g - 7.3378\Big\}$$

$$\left(\frac{\omega}{R}\right)_0 = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$

$$\left(\frac{\omega}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left(\frac{1}{2}\frac{a^2}{R^2}\right)\left\{J_0\left(\beta\frac{r}{a}\right) + \eta\right\}\frac{1}{\delta}$$

$\beta = 3.8317$

$\gamma = 0.4028$

$\delta = 1.4028$

$$\frac{1}{R}\frac{\partial\omega}{\partial r} = -r\left(\frac{1}{R^2}\right)\sin^2\theta - \frac{f}{\delta}\frac{1}{2}\frac{a^2}{R^2}\frac{\beta}{a}J_0'\left(\beta\frac{r}{a}\right)$$

$$\frac{1}{R}\frac{\partial\omega_0}{\partial r} = -r\left(\frac{1}{R^2}\right)\sin^2\theta$$

$$\frac{1}{R}\frac{\partial^2\omega}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta - \frac{f}{\delta}\frac{1}{2}\frac{a^2}{R^2}\frac{\beta^2}{a^2}J_0''\left(\beta\frac{r}{a}\right)$$

$$\frac{1}{r}\frac{\partial^2\omega_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta$$

$$-\left\{\frac{1}{r}\frac{\partial\omega}{\partial r}\frac{\partial^2\omega}{\partial r^2} - \frac{1}{r}\frac{\partial\omega_0}{\partial r}\frac{\partial^2\omega_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}(\sin^2\theta)^2 - \frac{1}{R^2}\left[\sin^2\theta + \frac{1}{2}\frac{f}{\delta}\frac{a\beta}{r}J_0'\left(\beta\frac{r}{a}\right)\right]\left[\sin^2\theta + \frac{1}{2}\frac{f}{\delta}\beta^2 J_0''\left(\beta\frac{r}{a}\right)\right]$$

$$= -\frac{1}{R^2}\left[\frac{1}{2}\frac{f}{\delta}\beta^2\sin^2\theta\left\{J_0'' + \frac{1}{\left(\beta\frac{r}{a}\right)}J_0'\right\} + \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r}J_0'J_0''\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2\omega}{\partial r^2}\frac{\partial^2\omega}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^2\omega_0}{\partial r^2}\frac{\partial^2\omega_0}{\partial\theta^2}\right\}$$

$$= -\frac{1}{R^2}\cos 2\theta \cdot \frac{1}{2}\frac{f}{\delta}\beta^2 J_0''$$

$$\nabla^4\phi = \frac{E}{R^2}\left[J_0 \frac{1}{2}\frac{f}{\delta}\beta^2 \sin^2\theta - \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r} J_0' J_0'' - \frac{1}{2}\frac{f}{\delta}\beta^2 J_0'' \cos 2\theta \right]$$ $\underline{\underline{362}}$

$$= \frac{E}{R^2}\left[J_0 \frac{1}{4}\frac{f}{\delta}\beta^2 (1-\cos 2\theta) - \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r} J_0' J_0'' - \frac{1}{2}\frac{f}{\delta}\beta^2 J_0'' \cos 2\theta \right]$$

$$= \frac{1}{4}\frac{f}{\delta}\beta^2 \frac{E}{R^2}\left[\left\{ J_0 - \beta^2 \frac{f}{\delta} \frac{J_0' J_0''}{(\beta \frac{r}{a})} \right\} - \cos 2\theta \left\{ J_0 + 2J_0'' \right\} \right]$$

$$= \frac{1}{4}\frac{gE}{R^2}\left[\left\{ J_0 - g \frac{J_0' J_0''}{z} \right\} - \cos 2\theta \left(J_0 + 2J_0'' \right) \right\}$$

where $\quad g = \frac{f}{\delta}\beta^2 \quad , \qquad z = (\beta \frac{r}{a})$

$$\frac{J_0' J_0''}{z} = \frac{J_1 J_1'}{z} = \frac{J_1^2}{z^2} - \frac{J_1 J_2}{z}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2)!\,(\frac{1}{2}z)^{2n}}{4(n!)(n+1)!(n+1)!(n+2)!} - \sum_{n=0}^{\infty} \frac{(-1)^n (2n+3)!\,(\frac{1}{2}z)^{2n+2}}{2(n!)(n+1)!(n+2)!(n+3)!}$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{z}{2}\right)^2 + \frac{5}{48}\left(\frac{z}{2}\right)^4 - \frac{7}{288}\left(\frac{z}{2}\right)^6 + \frac{7}{1920}\left(\frac{z}{2}\right)^8 - \frac{11}{28800}\left(\frac{z}{2}\right)^{10}$$

$$+ \frac{143}{4838400}\left(\frac{z}{2}\right)^{12} - \cdots\cdots$$

$$- \left\{ \frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{5}{24}\left(\frac{z}{2}\right)^4 + \frac{7}{96}\left(\frac{z}{2}\right)^6 - \frac{7}{480}\left(\frac{z}{2}\right)^8 + \frac{11}{5760}\left(\frac{z}{2}\right)^{10} + \frac{143}{806400}\left(\frac{z}{2}\right)^{12} \right\} -$$

Thus

$$\frac{J_0' J_0''}{z} = \frac{1}{4} - \frac{1}{2}\left(\frac{z}{2}\right)^2 + \frac{5}{16}\left(\frac{z}{2}\right)^4 - \frac{7}{72}\left(\frac{z}{2}\right)^6 + \frac{7}{384}\left(\frac{z}{2}\right)^8 - \frac{11}{4800}\left(\frac{z}{2}\right)^{10} + \frac{143}{691200}\left(\frac{z}{2}\right)^{12} - \cdots$$

The particular integral for this term is

$$\frac{\phi_1}{R^2} = -\frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{\delta}\right)^2 E \Big\{ \frac{1}{16}\left(\frac{z}{2}\right)^4 - \frac{1}{72}\left(\frac{z}{2}\right)^6 + \frac{5}{2304}\left(\frac{z}{2}\right)^8 - \frac{7}{28800}\left(\frac{z}{2}\right)^{10} + \frac{7}{345600}\left(\frac{z}{2}\right)^{12}$$
$$- \frac{11}{4\times 2116800}\left(\frac{z}{2}\right)^{14} + \frac{143}{691200\times 784\times 4}\left(\frac{z}{2}\right)^{16} - \cdots \Big\}$$

The particular integral for the term $\frac{1}{4}\frac{gE}{R^2}J_0$ is

$$\frac{\phi_2}{R^2} = \frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{\delta}\right)\frac{1}{\beta^2}J_0$$

The particular integral for the term $-\frac{1}{4}\frac{gE}{R^2}\cos 2\theta\,(J_0 + 2J_0'')$ is

$$\frac{\phi_3}{R^2} = -\frac{1}{4}\left(\frac{a}{R}\right)^4\left(\frac{f}{\delta}\right)\frac{1}{\beta^2}\cos 2\theta\; J_2$$

Then the total particular integral is

$$\frac{\phi_1+\phi_2+\phi_3}{R^2} = \frac{1}{4}\left(\frac{a}{R}\right)^4\frac{1}{\beta^4}\, gE \Big[\Big\{ J_0 - \frac{g}{64}\Big[\frac{1}{4}\left(\frac{z}{2}\right)^4 - \frac{1}{18}\left(\frac{z}{2}\right)^6 + \frac{5}{576}\left(\frac{z}{2}\right)^8 - \frac{7}{7200}\left(\frac{z}{2}\right)^{10}$$
$$+ \frac{7}{86400}\left(\frac{z}{2}\right)^{12} - \frac{11}{2116800}\left(\frac{z}{2}\right)^{14} + \frac{143}{691200\times 784}\left(\frac{z}{2}\right)^{16} - \cdots\Big]\Big\}$$
$$- J_2 \cos 2\theta \Big\} = \frac{\Phi}{R^2}$$

The stresses due to this particular integral are:

$$\frac{1}{r}\frac{\partial\Phi}{\partial r} = \left(\frac{\beta}{a}\right)^2 \frac{1}{z}\frac{\partial\Phi}{\partial z} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[\frac{J_0'}{z} - \frac{9}{\pi}\Big[\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{1}{12}\left(\frac{z}{2}\right)^4 + \frac{5}{288}\left(\frac{z}{2}\right)^6$$

$$- \frac{7}{2880}\left(\frac{z}{2}\right)^8 + \frac{7}{28800}\left(\frac{z}{2}\right)^{10} - \frac{11}{604800}\left(\frac{z}{2}\right)^{12} + \frac{143}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\Big]$$

$$- \frac{J_2'}{z}\cos 2\theta\Big\}$$

$$\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = \left(\frac{\beta}{a}\right)^2 \frac{1}{z^2}\frac{\partial^2\Phi}{\partial\theta^2} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[+\frac{J_2}{z^2}\,4\cos 2\theta\Big]$$

$$\widehat{rr}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[\frac{J_0'}{z} - \frac{9}{\pi}\Big\{\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{1}{12}\left(\frac{z}{2}\right)^4 + \frac{5}{288}\left(\frac{z}{2}\right)^6 - \frac{7}{2880}\left(\frac{z}{2}\right)^8 + \frac{7}{28800}\left(\frac{z}{2}\right)^{10}$$

$$- \frac{11}{604800}\left(\frac{z}{2}\right)^{12} + \frac{143}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\Big\} - \cos 2\theta\left(\frac{J_2'}{z} - \frac{4J_2}{z^2}\right)\Big]$$

$$\widehat{\theta\theta}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[J_0'' - \frac{9}{\pi}\Big[\frac{3}{4}\left(\frac{z}{2}\right)^2 - \frac{5}{12}\left(\frac{z}{2}\right)^4 + \frac{35}{288}\left(\frac{z}{2}\right)^6 - \frac{63}{2880}\left(\frac{z}{2}\right)^8 + \frac{22}{28800}\left(\frac{z}{2}\right)^{10}$$

$$- \frac{143}{604800}\left(\frac{z}{2}\right)^{12} + \frac{2145}{135475200}\left(\frac{z}{2}\right)^{14} - \cdots\Big] - J_2''\cos 2\theta\Big\}$$

$$\widehat{\theta r}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{gE}{\beta^2}\Big[2\sin 2\theta\left(\frac{J_2'}{z} - \frac{J_2}{z^2}\right)\Big]$$

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$$\frac{1}{E}(\widehat{rr} - \nu\widehat{\theta\theta}) = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{g}{\beta^2}\Big[\left(\frac{J_0'}{z} - \nu J_0''\right) - g\Big\{\frac{(1-3\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{(1-5\nu)}{48}\left(\frac{z}{2}\right)^4$$

$$+ \frac{5(1-7\nu)}{288\times 4}\left(\frac{z}{2}\right)^6 - \frac{7(1-9\nu)}{2880\times 4}\left(\frac{z}{2}\right)^8 + \frac{7(1-11\nu)}{4\times 28800}\left(\frac{z}{2}\right)^{10} - \frac{11(1-13\nu)}{604800\times 4}\left(\frac{z}{2}\right)^{12}$$

$$+ \frac{143(1-15\nu)}{4\times 135475200}\left(\frac{z}{2}\right)^{14} - - - - \Big\} - \cos 2\theta\left\{\frac{J_2'}{z} - \frac{4J_2}{z^2} - \nu J_2''\right\}\Big]$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{\partial w_2}{\partial r}\right)^2\right\} = \frac{1}{2}\Big[\frac{1}{2}\left(\frac{a}{R}\right)\beta\frac{A}{\delta}J_0'\left\{\frac{1}{2}\left(\frac{a}{R}\right)\beta\frac{A}{\delta}J_0' + 2\left(\frac{a}{R}\right)\sin^2\theta\right\}\Big]$$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\Big[\left(zJ_0' + g\frac{J_0'^2}{2}\right) - zJ_0'\cos 2\theta\Big]$$

$$J_0'^2 = J_1^2 = \sum_n \frac{(-1)^n(2n+2)!\left(\frac{z}{2}\right)^{2n+2}}{n!\,(n+2)!\,(n+1)!\,(n+1)!}$$

$$= \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^4 + \frac{5}{12}\left(\frac{z}{2}\right)^6 - \frac{7}{72}\left(\frac{z}{2}\right)^8 + - - -$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\Big[\Big\{\left(\frac{J_0'}{z} - zJ_0'\right) - \nu J_0''\Big\} - g\Big\{\frac{3(3-\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{5(5-\nu)}{48}\left(\frac{z}{2}\right)^4$$

$$+ \frac{5\times 7(7-\nu)}{1152}\left(\frac{z}{2}\right)^6 - \frac{7\times 9(9-\nu)}{11520}\left(\frac{z}{2}\right)^8 + \frac{7\times 11(11-\nu)}{115200}\left(\frac{z}{2}\right)^{10} - \frac{11\times 13(13-\nu)}{2419200}\left(\frac{z}{2}\right)^{12}$$

$$+ \frac{143\times 15(15-\nu)}{541900800}\left(\frac{z}{2}\right)^{14} - - - \Big\} - \cos 2\theta\left(\frac{J_2'}{z} - \frac{4J_2}{z^2} - \nu J_2'' - zJ_0'\right)\Big]$$

$$\frac{u_1}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{1}{\beta^3} g \left[\left\{ (\quad z J_0'' \quad) - \nu J_0' \right\} - g \left\{ \frac{(3-\nu)}{8}\left(\frac{z}{2}\right)^3 - \frac{(5-\nu)}{24}\left(\frac{z}{2}\right)^5 \right. \right.$$

$$+ \frac{5(7-\nu)}{576}\left(\frac{z}{2}\right)^7 - \frac{7(9-\nu)}{5760}\left(\frac{z}{2}\right)^9 + \frac{7(11-\nu)}{57600}\left(\frac{z}{2}\right)^{11} - \frac{11(13-\nu)}{1209600}\left(\frac{z}{2}\right)^{13}$$

$$\left. + \frac{143(15-\nu)}{270950400}\left(\frac{z}{2}\right)^{15} - \cdots \right\} - \cos 2\theta \left\{ -\nu J_2' - J_2' - z J_1' - J_0' \right\}$$

$$\int \left(\frac{J_2'}{z} - \frac{4 J_2}{z^2} - z J_0' \right) dz = \int \left\{ \frac{J_2'}{z} - \left(J_2'' + \frac{J_2'}{z} + J_2 \right) - z J_0' \right\} dz$$

$$= -J_2' - \int (J_2 + z J_0') \, dz$$

$$= -J_2' - \int \left\{ \frac{2J_1}{z} - J_0(z) + z J_0' \right\} dz = -J_2' + \int \left(\frac{J_0'}{z} - z J_0' \right) dz$$

$$+ \int \left(\frac{J_0'}{z} + J_0 \right) dz = -J_2' + z J_0'' + \int \left(\frac{J_0'}{z} + J_0 \right) dz = -J_2' + z J_0'' - J_0'$$

But
$$J_0 = J_2 - 2 J_0''$$
$$\frac{J_0'}{z} = -J_2 + J_0''$$
$$J_0 + \frac{J_0'}{z} = -J_0''$$

$$\frac{u}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{g}{\beta^2} \left[\left\{ J_0'' - \nu \frac{J_0'}{z} \right\} - g \left\{ \frac{(3-\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{(5-\nu)}{48}\left(\frac{z}{2}\right)^4 + \cdots \right\} \right.$$

$$\left. + \cos 2\theta \left\{ \frac{J_2'}{z} + J_1' + \frac{J_0'}{z} + \nu \frac{J_2'}{z} \right\} \right]$$

$$-\frac{u}{r}+\frac{1}{E}\left(\widehat{\theta\theta}_1-\widehat{rr}_1\right)=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{-J_2''+\frac{\nu J_2'}{z}-\frac{4\nu J_2}{z^2}-\frac{J_2'}{z}-J_1'-\frac{J_0'}{z}-\frac{\nu J_2''}{z}\right\}\right]$$ 367

$$=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{-J_2''-\frac{J_2'}{z}-J_1'+\frac{J_1}{z}-\frac{4\nu J_2}{z^2}\right\}\right]$$

$$=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{J_2-\frac{4(1+\nu)J_2}{z^2}+\frac{J_1}{z}-J_1'\right\}\right]$$

$$\boxed{\frac{V}{R}=\frac{1}{4}\left(\frac{a}{R}\right)^3\frac{g}{\beta^3}\left[\frac{\sin 2\theta}{2}\left\{-zJ_2''-J_2'-zJ_1'+J_1-\frac{4\nu J_2}{z}\right\}\right]}$$

The total stress component can be expressed as

$$\widehat{rr}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\frac{1}{2}Q_0-\frac{J_1}{z}-\frac{g}{4}\left\{\frac{1}{4}\left(\frac{z}{2}\right)^2-\frac{1}{12}\left(\frac{z}{2}\right)^4+\frac{5}{288}\left(\frac{z}{2}\right)^6-\frac{7}{2880}\left(\frac{z}{2}\right)^8+\frac{7}{28800}\left(\frac{z}{2}\right)^{10}\right.\right.$$
$$\left.\left.-\frac{11}{604800}\left(\frac{z}{2}\right)^{12}+\frac{143}{135475200}\left(\frac{z}{2}\right)^{14}-\cdots\right\}-\cos 2\theta\left\{2P_2+\frac{J_1}{z}-\frac{6J_2}{z^2}\right\}\right]$$

$$\widehat{\theta\theta}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\frac{1}{2}Q_0-\frac{1}{2}(J_0-J_2)-\frac{g}{4}\left\{\frac{3}{4}\left(\frac{z}{2}\right)^2-\frac{5}{12}\left(\frac{z}{2}\right)^4+\frac{35}{288}\left(\frac{z}{2}\right)^6-\frac{63}{2880}\left(\frac{z}{2}\right)^8\right.\right.$$
$$\left.+\frac{27}{28800}\left(\frac{z}{2}\right)^{10}-\frac{143}{604800}\left(\frac{z}{2}\right)^{12}+\frac{2145}{135475200}\left(\frac{z}{2}\right)^{14}-\cdots\right]+\cos 2\theta\left\{2P_2+12R_2z^2\right.$$
$$\left.\left.-\left(\frac{6}{z^2}-1\right)J_2+\frac{J_1}{z}\right\}\right]$$

$$\widehat{r\theta}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\sin 2\theta\left\{2P_2+6R_2z^2-\frac{6J_2}{z^2}+\frac{9J_1}{z}\right\}\right]$$

The total deflection is

$$\frac{u}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{g}{\beta^3}\Big[\frac{(1-\nu)}{2} Q_0 z + J_1 - z J_0 + \nu J_1 - \frac{g}{4}\Big\{ \frac{(3-\nu)}{2}\left(\frac{z}{2}\right)^3 - \frac{(5-\nu)}{6}\left(\frac{z}{2}\right)^5$$

$$+ \frac{5(7-\nu)}{144}\left(\frac{z}{2}\right)^7 - \frac{7(9-\nu)}{1440}\left(\frac{z}{2}\right)^9 + \frac{7(11-\nu)}{14400}\left(\frac{z}{2}\right)^{11} - \frac{11(13-\nu)}{302400}\left(\frac{z}{2}\right)^{13}$$

$$+ \frac{143(15-\nu)}{67737600}\left(\frac{z}{2}\right)^{15} - \cdots \Big\} - \cos 2\theta \Big\{ 2(1+\nu) P_2 z + 4\nu R_2 z^3 + J_1 - z J_1'$$

$$- (1+\nu) J_2' \Big\} \Big]$$

$$\frac{v}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{g}{\beta^3}\Big[\sin 2\theta \Big\{ 2(1+\nu) P_2 z + 6(1+\nu) R_2 z^3 - \frac{z J_2''}{2} - \frac{J_2'}{2} - \frac{z J_1'}{2}$$

$$+ \frac{1}{2} J_1 - \frac{2\nu J_2}{z} \Big\} \Big]$$

At $r = a$, $z = \beta$ $\qquad \frac{\beta}{2} = \frac{3.8317}{2} = 1.6159$

$$w_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{qE}{t\beta^2}\Big[\frac{1}{2} Q_0 - \frac{g}{4}\Big\{ 0.25 \times 1.6159^2 - 0.083333 \times 1.6159^4 + 0.00173611 \times 1.6159^6$$

$$- 0.0002430556 \times 1.6159^8 + 0.0000243056 \times 1.6159^{10} - 0.00000181878 \times 1.6159^{12}$$

$$+ 0.0000010555 \times 1.6159^{14} - \cdots \Big\} - \cos 2\theta \Big\{ 2P_2 - 6\,\frac{0.4025}{3.8317^2} \Big\} \Big]$$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{qE}{\beta^2}\Big[\frac{1}{2} Q_0 - \frac{g}{4}\Big\{ 0.25 \times 2.6111 - 0.083333 \times 6.8178 + 0.00173611 \times 17.8020$$

$$- 0.0002430556 \times 46.4828 + 0.0000243056 \times 121.37 - 0.00000181878 \times 316.91$$

$$+ 0.0000010555 \times 827.48 - \cdots \Big\} - \cos 2\theta \Big\{ 2P_2 - 0.1645 \Big\} \Big]$$

+0.65278		+1.95834	
−0.56815		−2.84075	
+0.03091		+0.21637	
−0.01130		−0.10170	
+0.00295		+0.03245	
−0.00576		−0.07484	
+0.00087	0.100	+0.01305	−0.824

$$\widehat{rr}_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[\frac{1}{2}Q_0 - 0.02509 - \cos 2\theta\left(2P_2 - 0.1645\right)\right]$$

$$\widehat{\theta\theta}_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[\frac{1}{2}Q_0 + 0.4027 + 0.2069 + \cos 2\theta\left\{2P_2 + 176.18\,R_2 + 0.2340\right\}\right]$$

$$\widehat{r\theta}_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[\sin 2\theta\left\{2P_2 + 88.09\,R_2 - 0.1145\right\}\right]$$

The non-uniform portion of $\frac{u}{R_a}$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[-\cos 2\theta\left\{2(1+\nu)P_2 + 58.7276\,\nu\,R_2 + 0.4027 + (1+\nu)0.05483\right\}\right]$$

$$\frac{v}{R_a} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2}\left[\sin 2\theta\left\{2(1+\nu)P_2 + 88.0914(1+\nu)R_2 + 0.3478 - 0.054838\right\}\right]$$

Put $\quad \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\rho E}{\beta^2} = \eta$

Then the stress + displacement conditions give

$$\frac{1}{2} + r_0 = \eta g \left\{ \frac{1}{2} Q_0 - 0.0250 g \right\} \qquad (1)$$

$$\frac{1}{2} - r_0 = \eta g \left\{ \frac{1}{2} Q_0 + 0.4027 + 0.206 g \right\} \qquad (2)$$

$$(3) \quad \frac{1}{2} - 6\frac{q}{r^2} - 4S_2 = \eta g \left\{ 0.1645 - 2P_2 \right\}$$

$$(4) \quad 6\frac{q}{r^2} - \frac{1}{2} = \eta g \left\{ 2P_2 + 176.18 P_2 + 0.2360 \right\}$$

$$(5) \quad \frac{1}{2} + 6\frac{q}{r^2} + 2S_2 = \eta g \left\{ 0.1645 - 2P_2 - 8[illegible] P_2 \right\}$$

$$(6) \quad \frac{1}{2}(1+\lambda) + 2(1+\lambda)\frac{q}{r^2} + 4S_2 = \eta g \left\{ -2(1+\lambda)P_2 - 58.722(1+\lambda)P_2 - 0.4022(1+\lambda) \right\}$$

0.0548

$$(7) \quad 2(1+\lambda)\frac{q}{r^2} - \frac{1}{2}(1+\lambda) = \eta g \left\{ 2(1+\lambda)P_2 + 88.0[illegible](1+\lambda)P_2 + 0.3[illegible] - 0.05[illegible] \right\}$$

---

from (1) + (2)

$$1 = \eta g \left\{ Q_0 + 0.4027 + 0.181 g \right\}$$

$$\boxed{Q_0 = \frac{1}{\eta g} - (0.4027 + 0.181 g)}$$

$$r_0 = \eta g \left\{ -(0.20135 + 0.0905 g) - 0.[illegible] g \right\} = -\eta g \left\{ 0.20135 + 0.155 g \right\}$$

$$\boxed{r_0 = -\eta g\,(0.204 + 0.155 g)}$$

$$q_2 + 0.6667\,S_2 - 0.08333 = \eta g\left\{0.3333\,P_2 - 0.02742\right\}$$
$$q_2 + 0 - 0.08333 = \eta g\left\{0.3333\,P_2 + 29.36\,R_2 + 0.03967\right\}$$
$$q_2 + 0.3333\,S_2 + 0.08333 = \eta g\left\{-0.3333\,P_2 - 14.68\,R_2 + 0.02742\right\}$$
$$q_2 + 1.5385\,S_2 + 0.2500 = \eta g\left\{-P_2 - 6.7760\,R_2 - 0.1823\right\}$$
$$q_2 + 0 - 0.2500 = \eta g\left\{P_2 + 44.0457\,R_2 + 0.1275\right\}$$

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$$0.6667\,S_2 + 0 = \eta g\left\{-29.36\,R_2 - 0.06709\right\}$$
$$0.3333\,S_2 + 0.1667 = \eta g\left\{-0.6667\,P_2 - 44.04\,R_2 - 0.01225\right\}$$
$$1.2052\,S_2 + 0.1667 = \eta g\left\{-0.6667\,P_2 + 7.904\,R_2 - 0.2097\right\}$$
$$1.5385\,S_2 + 0.5000 = \eta g\left\{-2\,P_2 - 50.83\,R_2 - 0.3098\right\}$$

$$S_2 + 0 = \eta g\left\{-44.04\,P_2 - 0.1006\right\}$$
$$S_2 + 0.5000 = \eta g\left\{-2\,P_2 - 132.12\,R_2 - 0.03675\right\}$$
$$S_2 + 0.1383 = \eta g\left\{-0.5532\,P_2 + 6.558\,R_2 - 0.1740\right\}$$
$$S_2 + 0.3250 = \eta g\left\{-1.3\,P_2 - 33.038\,R_2 - 0.2014\right\}$$

$$0.5000 = \eta g\left\{-2\,P_2 - 88.08\,R_2 + 0.06365\right\}$$
$$0.3617 = \eta g\left\{-1.4468\,P_2 - 138.68\,R_2 + 0.1372\right\}$$
$$0.1867 = \eta g\left\{-0.7466\,P_2 - 37.596\,R_2 - 0.0274\right\}$$

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For the circular region,

$$\left(\frac{w}{R}\right)_0 = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ 1 - \left(\frac{r}{a}\right)^2 \sin^2\theta \right\}$$

$$\left(\frac{w}{R}\right) = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ 1 - \left(\frac{r}{a}\right)^2 \sin^2\theta - f_1\left[1 - \left(\frac{r}{a}\right)^2\right]^2 - f_2\left[1 - \left(\frac{r}{a}\right)^3\right]^2 - f_3\left[1 - \left(\frac{r}{a}\right)^4\right]^2 \right\}$$

where f_1, f_2, f_3, are amplitudes.

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{2}\frac{1}{R}\left\{ - 2\sin^2\theta + 4f_1\left[1 - \left(\frac{r}{a}\right)^2\right] + 6f_2\left[1 - \left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 8f_3\left[1 - \left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)_0 = \frac{1}{2}\frac{1}{R}\left\{ -2 \sin^2\theta \right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{ -2\sin^2\theta + 4f_1\left[1 - 3\left(\frac{r}{a}\right)^2\right] + 6f_2\left[2 - 5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 8f_3\left[3 - 7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{ -2\sin^2\theta \right\}$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R}\left\{ -\cos 2\theta \right\}$$

$\cancel{8} \times \cancel{8} \times 3^2$

$\frac{32^2}{14} = \frac{32 \times 16}{7}$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$=\frac{1}{R^2}(\sin^2\theta)^2-\frac{1}{R^2}\left\{\sin^2\theta-2f_1\left[1-\left(\frac{r}{a}\right)^2\right]-3f_2\left[1-\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right)-4f_3\left[1-\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}\left\{\sin^2\theta-2f_1\left[1-3\left(\frac{r}{a}\right)^2\right]-3f_2\left[2-5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right)-4f_3\left[3-7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}$$

$$=\frac{1}{R^2}\left\{2f_1\left[1-2\left(\frac{r}{a}\right)^2\right]+3f_2\left[1.5-3\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right)+4f_3\left[2-4\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}\underline{(1-\cos 2\theta)}$$

$$-\frac{1}{R^2}\left\{4f_1^2\left[1-4\left(\frac{r}{a}\right)^2+3\left(\frac{r}{a}\right)^4\right]+6f_1f_2\left[1-3\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^3+3\left(\frac{r}{a}\right)^5\right]\left(\frac{r}{a}\right)+8f_1f_3\left[1-3\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^4+3\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2\right.$$

$$+6f_1f_2\left[2-2\left(\frac{r}{a}\right)^2-5\left(\frac{r}{a}\right)^3+5\left(\frac{r}{a}\right)^5\right]\left(\frac{r}{a}\right)+8f_1f_3\left[3-3\left(\frac{r}{a}\right)^2-7\left(\frac{r}{a}\right)^4+7\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2$$

$$+9f_2^2\left[2-7\left(\frac{r}{a}\right)^3+5\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2+12f_2f_3\left[2-5\left(\frac{r}{a}\right)^3-2\left(\frac{r}{a}\right)^4+5\left(\frac{r}{a}\right)^7\right]\left(\frac{r}{a}\right)^3$$

$$\left.+12f_2f_3\left[3-3\left(\frac{r}{a}\right)^3-7\left(\frac{r}{a}\right)^4+7\left(\frac{r}{a}\right)^7\right]\left(\frac{r}{a}\right)^3+16f_3^2\left[3-10\left(\frac{r}{a}\right)^4+7\left(\frac{r}{a}\right)^8\right]\left(\frac{r}{a}\right)^4\right\}$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\}$$

$$=\frac{1}{R^2}\cos 2\theta\left\{2f_1\left[1-3\left(\frac{r}{a}\right)^2\right]+3f_2\left[2-5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right)+4f_3\left[3-7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}$$

The terms for the particular integral is then, <u>multiplied by</u> $\underline{R^2}$,

$$2f_1\left[1-2\left(\tfrac{r}{a}\right)^2\right] + 3f_2\left[1.5-3\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right) + 4f_3\left[2-4\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2$$

$$-4f_1^2\left[1-4\left(\tfrac{r}{a}\right)^2+3\left(\tfrac{r}{a}\right)^4\right] - 6f_1f_2\left[3-5\left(\tfrac{r}{a}\right)^2-6\left(\tfrac{r}{a}\right)^3+8\left(\tfrac{r}{a}\right)^5\right]\left(\tfrac{r}{a}\right) - 8f_1f_3\left[4-6\left(\tfrac{r}{a}\right)^2-8\left(\tfrac{r}{a}\right)^4+10\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2$$

$$-9f_2^2\left[2-7\left(\tfrac{r}{a}\right)^3+5\left(\tfrac{r}{a}\right)^6\right]\left(\tfrac{r}{a}\right)^2 - 12f_2f_3\left[5-8\left(\tfrac{r}{a}\right)^3-9\left(\tfrac{r}{a}\right)^4+12\left(\tfrac{r}{a}\right)^7\right]\left(\tfrac{r}{a}\right)^3 - 16f_3^2\left[3-10\left(\tfrac{r}{a}\right)^4+7\left(\tfrac{r}{a}\right)^8\right]\left(\tfrac{r}{a}\right)^4$$

$$-\cos 2\theta\left\{2f_1\left(\tfrac{r}{a}\right)^2 - 3f_2\left[0.5-2\left(\tfrac{r}{a}\right)^3\right]\left(\tfrac{r}{a}\right) - 4f_3\left[1-3\left(\tfrac{r}{a}\right)^4\right]\left(\tfrac{r}{a}\right)^2\right\}$$

$$= 2f_1(1-2f_1) + (4.5f_2 - 18f_1f_2)\left(\tfrac{r}{a}\right) + (-4f_1 + 8f_3 + 16f_1^2 - 32f_1f_3 - 18f_2^2)\left(\tfrac{r}{a}\right)^2$$

$$+(30f_1f_2 - 60f_2f_3)\left(\tfrac{r}{a}\right)^3 + (-9f_2 - 12f_1^2 + 36f_1f_2 + 48f_1f_3 - 48f_3^2)\left(\tfrac{r}{a}\right)^4$$

$$+(63f_2^2)\left(\tfrac{r}{a}\right)^5 + (-16f_3 - 48f_1f_2 + 64f_1f_3 + 96f_2f_3)\left(\tfrac{r}{a}\right)^6 + (108f_2f_3)\left(\tfrac{r}{a}\right)^7 + (-80f_1f_3 - 45f_2^2 + 160f_3^2)\left(\tfrac{r}{a}\right)^8$$

$$+(-144f_2f_3)\left(\tfrac{r}{a}\right)^{10} + -112f_3^2\left(\tfrac{r}{a}\right)^{12}$$

$$+\cos 2\theta\left\{1.5f_2\left(\tfrac{r}{a}\right) + (4f_3 - 2f_1)\left(\tfrac{r}{a}\right)^2 - 6f_2\left(\tfrac{r}{a}\right)^4 - 12f_3\left(\tfrac{r}{a}\right)^6\right\}$$

$$\frac{\Phi}{r^2} = E\left(\frac{a}{R}\right)^4 \Bigg[\frac{1}{4} q_0 \left(\frac{r}{a}\right)^2 + \frac{f_1}{32}(1-2f_1)\left(\frac{r}{a}\right)^4 + 0.02\, f_2 (1-4f_1)\left(\frac{r}{a}\right)^5 + \frac{1}{144}\left(2f_3 - f_1 + 4f_1^2 - 8f_1 f_3 - 4.5 f_2^2\right)\left(\frac{r}{a}\right)^6$$

$$+ \frac{6}{245} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^7 + \frac{1}{768}\left(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{63} f_2^2 \left(\frac{r}{a}\right)^9$$

$$+ \frac{1}{400}\left(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2\right)\left(\frac{r}{a}\right)^{10} + \frac{12}{1089} f_2 f_3 \left(\frac{r}{a}\right)^{11} + \frac{1}{2880}\left(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2\right)\left(\frac{r}{a}\right)^{12}$$

$$- \frac{1}{196} f_2 f_3 \left(\frac{r}{a}\right)^{14} - \frac{1}{448} f_3^2 \left(\frac{r}{a}\right)^{16} + \cos 2\theta \Big\{ \frac{1}{70} f_2 \left(\frac{r}{a}\right)^5 + \frac{1}{192}(2f_3 - f_1)\left(\frac{r}{a}\right)^6 - \frac{1}{320} f_2 \left(\frac{r}{a}\right)^8$$

$$- \frac{1}{480} f_3 \left(\frac{r}{a}\right)^{10} + p_2 \left(\frac{r}{a}\right)^2 + r_2 \left(\frac{r}{a}\right)^4 \Big\} \Bigg]$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial r} = E\left(\frac{a}{R}\right)^2 \Bigg[\frac{1}{2} q_0 + \frac{f_1}{8}(1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{1}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{1}{24}\left(2f_3 - f_1 + 4f_1^2 - 8 f_1 f_3 - 4.5 f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+ \frac{6}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{1}{96}\left(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3 f_2 - 4 f_1^2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{7} f_2^2 \left(\frac{r}{a}\right)^7$$

$$+ \frac{1}{40}\left(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2\right)\left(\frac{r}{a}\right)^8 + \frac{12}{99} f_2 f_3 \left(\frac{r}{a}\right)^9 + \frac{1}{240}\left(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$- \frac{1}{14} f_2 f_3 \left(\frac{r}{a}\right)^{12} - \frac{1}{28} f_3^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{1}{14} f_2 \left(\frac{r}{a}\right)^3 + \frac{1}{32}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{1}{40} f_2 \left(\frac{r}{a}\right)^6$$

$$- \frac{1}{48} f_3 \left(\frac{r}{a}\right)^8 + 2 p_2 + 4 r_2 \left(\frac{r}{a}\right)^2 \Big\} \Bigg]$$

$$
\begin{aligned}
\widehat{rr} = E\left(\frac{a}{R}\right)^2 \Big[& \frac{1}{2} q_0 + \frac{f_1}{8}(1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{1}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{1}{24}\left(2f_3 - f_1 + 4f_1^2 - 8f_1 f_3 - 4.5 f_2^2\right)\left(\frac{r}{a}\right)^4 \\
& + \frac{6}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{1}{96}\left(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3 f_2 - 4 f_1^2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{7} f_2^2 \left(\frac{r}{a}\right)^7 \\
& + \frac{1}{40}\left(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2\right)\left(\frac{r}{a}\right)^8 + \frac{12}{99} f_2 f_3 \left(\frac{r}{a}\right)^9 + \frac{1}{240}\left(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2\right)\left(\frac{r}{a}\right)^{10} \\
& - \frac{1}{14} f_2 f_3 \left(\frac{r}{a}\right)^{12} - \frac{1}{28} f_3^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{1}{70} f_2 \left(\frac{r}{a}\right)^3 + \frac{1}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{1}{80} f_2 \left(\frac{r}{a}\right)^6 - \frac{1}{80} f_3 \left(\frac{r}{a}\right)^8 \\
& - 2 p_2 \Big\} \Big]
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2 \Big[& \frac{1}{2} q_0 + \frac{3}{8} f_1 (1-2f_1)\left(\frac{r}{a}\right)^2 + \frac{4}{10} f_2 (1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{5}{24}\left(2f_3 - f_1 + 4f_1^2 - 8 f_1 f_3 - 4.5 f_2^2\right)\left(\frac{r}{a}\right)^4 \\
& + \frac{36}{35} f_2 (f_1 - 2f_3)\left(\frac{r}{a}\right)^5 + \frac{7}{96}\left(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3 f_2 - 4 f_1^2\right)\left(\frac{r}{a}\right)^6 + \frac{8}{7} f_2^2 \left(\frac{r}{a}\right)^7 \\
& + \frac{9}{40}\left(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2\right)\left(\frac{r}{a}\right)^8 + \frac{120}{99} f_2 f_3 \left(\frac{r}{a}\right)^9 + \frac{11}{240}\left(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2\right)\left(\frac{r}{a}\right)^{10} \\
& - \frac{13}{14} f_2 f_3 \left(\frac{r}{a}\right)^{12} - \frac{15}{28} f_3^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ \frac{20}{70} f_2 \left(\frac{r}{a}\right)^3 + \frac{15}{96}(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{14}{80} f_2 \left(\frac{r}{a}\right)^6 \\
& - \frac{15}{80} f_3 \left(\frac{r}{a}\right)^8 + 2 p_2 + 12 r_2 \left(\frac{r}{a}\right)^2 \Big\} \Big]
\end{aligned}
$$

$$\widehat{r\theta} = E\left(\frac{a}{R}\right)^2\left[\sin 2\theta\left\{\frac{1}{70}f_2\left(\frac{r}{a}\right)^3+\frac{5}{96}(2f_3-f_1)\left(\frac{r}{a}\right)^4-\frac{3.5}{80}f_2\left(\frac{r}{a}\right)^6-\frac{3}{80}f_3\left(\frac{r}{a}\right)^8+2p_2+6r_2\left(\frac{r}{a}\right)^2\right\}\right]$$

$$\widehat{rr}-\nu\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2\left[\frac{1}{2}(1-\nu)q_0+\frac{(1-3\nu)}{8}f_1(1-2f_1)\left(\frac{r}{a}\right)^2+\frac{(1-4\nu)}{10}f_2(1-4f_1)\left(\frac{r}{a}\right)^3+\frac{(1-5\nu)}{24}(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2)\left(\frac{r}{a}\right)^4\right.$$

$$+\frac{6(1-6\nu)}{35}f_2(f_1-2f_3)\left(\frac{r}{a}\right)^5+\frac{(1-7\nu)}{96}(12f_1f_2^2+16f_1f_3-16f_3^2-3f_2^2-4f_1^2)\left(\frac{r}{a}\right)^6+\frac{(1-8\nu)}{7}f_2^2\left(\frac{r}{a}\right)^7$$

$$+\frac{(1-9\nu)}{40}(4f_1f_3^2+6f_1f_3-f_3-3f_1f_2^2)\left(\frac{r}{a}\right)^8+\frac{12(1-10\nu)}{99}f_2f_3\left(\frac{r}{a}\right)^9+\frac{(1-11\nu)}{240}(32f_3^2-16f_1f_3-9f_2^2)\left(\frac{r}{a}\right)^{10}$$

$$-\frac{(1-13\nu)}{14}f_2f_3\left(\frac{r}{a}\right)^{12}-\frac{(1-15\nu)}{28}f_3^2\left(\frac{r}{a}\right)^{14}+\cos 2\theta\left\{\frac{(1-20\nu)}{70}f_2\left(\frac{r}{a}\right)^3+\frac{(1-15\nu)}{96}(2f_3-f_1)\left(\frac{r}{a}\right)^4-\frac{(1-14\nu)}{80}f_2\left(\frac{r}{a}\right)^6\right.$$

$$\left.\left.-\frac{(1-15\nu)}{80}f_3\left(\frac{r}{a}\right)^8-2(1+\nu)p_2-12\nu r_2\left(\frac{r}{a}\right)^2\right\}\right]$$

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$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2 = \frac{1}{2}\left(\frac{a}{R}\right)^2 \Big[(\cos 2\theta - 1)\Big\{ 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] + 3f_2\left[1-\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[1-\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2 \Big\}$$

$$+ \Big\{ 4f_1^2\left[1-2\left(\frac{r}{a}\right)^2+\left(\frac{r}{a}\right)^4\right] + 12f_1f_2\left[1-\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^3+\left(\frac{r}{a}\right)^5\right]\left(\frac{r}{a}\right) + 16f_1f_3\left[1-\left(\frac{r}{a}\right)^2-\left(\frac{r}{a}\right)^4+\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2$$

$$+ 9f_2^2\left[1-2\left(\frac{r}{a}\right)^3+\left(\frac{r}{a}\right)^6\right]\left(\frac{r}{a}\right)^2 + 24f_2f_3\left[1-\left(\frac{r}{a}\right)^3-\left(\frac{r}{a}\right)^4+\left(\frac{r}{a}\right)^7\right]\left(\frac{r}{a}\right)^3 + 16f_3^2\left[1-2\left(\frac{r}{a}\right)^4+\left(\frac{r}{a}\right)^8\right]\left(\frac{r}{a}\right)^4\Big\}\Big]$$

$$= \frac{1}{2}\left(\frac{a}{R}\right)^2\Big[\cos 2\theta \Big\{ 2f_1\left(\frac{r}{a}\right)^2 + 3f_2\left(\frac{r}{a}\right)^3 + 2(2f_3 - f_1)\left(\frac{r}{a}\right)^4 - 3f_2\left(\frac{r}{a}\right)^6 - 4f_3\left(\frac{r}{a}\right)^8 \Big\}$$

$$+ 2f_1(2f_1-1)\left(\frac{r}{a}\right)^2 + (12f_1f_2 - 3f_2)\left(\frac{r}{a}\right)^3 + (-8f_1^2 + 16f_1f_3 + 9f_2^2 - 4f_3 + 2f_1)\left(\frac{r}{a}\right)^4$$

$$+ (-12f_1f_2 + 24f_2f_3)\left(\frac{r}{a}\right)^5 + (4f_1^2 - 12f_1f_2 - 16f_1f_3 + 16f_3^2 + 3f_2)\left(\frac{r}{a}\right)^6$$

$$- 18f_2^2\left(\frac{r}{a}\right)^7 + (12f_1f_2 - 16f_1f_3 \overset{-4f_3}{-} 24f_2f_3)\left(\frac{r}{a}\right)^8 + (-24f_2f_3)\left(\frac{r}{a}\right)^9 + (16f_1f_3 + 9f_2^2 - 32f_3^2)\left(\frac{r}{a}\right)^{10}$$

$$+ 24f_2f_3\left(\frac{r}{a}\right)^{12} + 16f_3^2\left(\frac{r}{a}\right)^{14}\Big\}\Big]$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial v_0}{\partial r}\right)^2=\left(\frac{a}{R}\right)^2\Big[\cos 2\theta\Big\{f_1\left(\frac{r}{a}\right)^2+\frac{3}{2}f_2\left(\frac{r}{a}\right)^3+(2f_3-f_1)\left(\frac{r}{a}\right)^4-\frac{3}{2}f_2\left(\frac{r}{a}\right)^6-2f_3\left(\frac{r}{a}\right)^8\Big\}$$

$$-f_1(1-2f_1)\left(\frac{r}{a}\right)^2-\frac{3}{2}f_2(1-4f_2)\left(\frac{r}{a}\right)^3-\left(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$-6f_2(f_1-2f_3)\left(\frac{r}{a}\right)^5+\frac{1}{2}\left(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2\right)\left(\frac{r}{a}\right)^6-9f_2^2\left(\frac{r}{a}\right)^7$$

$$-2\left(4f_1f_3+6f_2f_3-f_3-3f_1f_2\right)\left(\frac{r}{a}\right)^8-12f_2f_3\left(\frac{r}{a}\right)^9-\frac{1}{2}\left(32f_3^2-16f_1f_3-9f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+12f_2f_3\left(\frac{r}{a}\right)^{12}+8f_3^2\left(\frac{r}{a}\right)^{14}\Big\}\Big]$$

$$\frac{\partial u}{\partial r}=\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}(1-\nu)q_0+\frac{3(3-\nu)}{8}f_1(1-2f_1)\left(\frac{r}{a}\right)^2+\frac{4(4-\nu)}{10}f_2(1-4f_1)\left(\frac{r}{a}\right)^3$$

$$+\frac{5(5-\nu)}{24}\left(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2\right)\left(\frac{r}{a}\right)^4+\frac{36(6-\nu)}{35}f_2(f_1-2f_3)\left(\frac{r}{a}\right)^5$$

$$+\frac{7(7-\nu)}{96}\left(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2\right)\left(\frac{r}{a}\right)^6+\frac{8(8-\nu)}{7}f_2^2\left(\frac{r}{a}\right)^7$$

$$+\frac{9(9-\nu)}{40}\left(4f_1f_3+6f_2f_3-f_3-3f_1f_2\right)\left(\frac{r}{a}\right)^8+\frac{120(10-\nu)}{99}f_2f_3\left(\frac{r}{a}\right)^9+\frac{11(11-\nu)}{240}\left(32f_3^2-16f_1f_3-9f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$-\frac{13(13-\nu)}{14}f_2f_3\left(\frac{r}{a}\right)^{12}-\frac{15(15-\nu)}{28}f_3^2\left(\frac{r}{a}\right)^{14}$$

$$-\cos 2\theta \Big\{ f_1\left(\frac{r}{a}\right)^2 + \frac{(104+90\nu)}{70} f_2\left(\frac{r}{a}\right)^3 + \frac{(95+15\nu)}{96}(2f_3-f_1)\left(\frac{r}{a}\right)^4 - \frac{(119+14\nu)}{80} f_2\left(\frac{r}{a}\right)^6 - \frac{(159+15\nu)}{80} f_3\left(\frac{r}{a}\right)^8$$

$$+ 2(1+\nu)\beta_2 + 12\nu r_2\left(\frac{r}{a}\right)^2\Big\}\Big]$$

$$\frac{u}{R} = \left(\frac{a}{R}\right)^3\Big[\frac{1}{2}(1-\nu)q\left(\frac{a}{r}\right) + \frac{(3-\nu)}{8} f_1(1-2f_1)\left(\frac{r}{a}\right)^3 + \frac{(4-\nu)}{10} f_2(1-4f_1)\left(\frac{r}{a}\right)^3 + \frac{(5-\nu)}{24}(2f_3-f_1+4f_1^2-8f_1f_3-4.5f_2^2)\left(\frac{r}{a}\right)^5$$

$$+ \frac{6(6-\nu)}{35} f_2(f_1-2f_3)\left(\frac{r}{a}\right)^6 + \frac{(7-\nu)}{96}(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2)\left(\frac{r}{a}\right)^7 + \frac{(8-\nu)}{7} f_2^2\left(\frac{r}{a}\right)^8$$

$$+ \frac{(9-\nu)}{40}(4f_1f_3+6f_2f_3-f_3-3f_1f_2)\left(\frac{r}{a}\right)^9 + \frac{12(10-\nu)}{91} f_2f_3\left(\frac{r}{a}\right)^{10} + \frac{(11-\nu)}{240}(32f_3^2-16f_1f_3-9f_2^2)\left(\frac{r}{a}\right)^{11}$$

$$- \frac{(13-\nu)}{14} f_2f_3\left(\frac{r}{a}\right)^{13} - \frac{(15-\nu)}{28} f_3^2\left(\frac{r}{a}\right)^{15}$$

$$-\cos 2\theta\Big\{\frac{1}{3} f_1\left(\frac{r}{a}\right)^3 + \frac{(26+5\nu)}{70} f_2\left(\frac{r}{a}\right)^4 + \frac{(19+3\nu)}{96}(2f_3-f_1)\left(\frac{r}{a}\right)^5 - \frac{(17+2\nu)}{80} f_2\left(\frac{r}{a}\right)^7 - \frac{(53+5\nu)}{240} f_3\left(\frac{r}{a}\right)^9$$

$$+ 2(1+\nu)\beta_2\left(\frac{r}{a}\right) + 4\nu r_2\left(\frac{r}{a}\right)^3\Big\}\Big]$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \left(\frac{a}{R}\right)^2 \left[\cos 2\theta \left\{ \frac{1}{3} f_1 \left(\frac{r}{a}\right)^2 + \frac{(23+2\nu)}{35} f_2 \left(\frac{r}{a}\right)^3 + \frac{(17+\nu)}{48} (2f_3 - f_1)\left(\frac{r}{a}\right)^4 - \frac{(31+\nu)}{80} f_2 \left(\frac{r}{a}\right)^6 \right.\right.$$

$$\left.\left. - \frac{(49+\nu)}{120} f_3 \left(\frac{r}{a}\right)^8 + 4(1+\nu) p_2 + 4(3+\nu) r_2 \left(\frac{r}{a}\right)^2 \right\}\right]$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \left[\sin 2\theta \left\{ \frac{1}{6} f_1 \left(\frac{r}{a}\right)^3 + \frac{(23+2\nu)}{70} f_2 \left(\frac{r}{a}\right)^4 + \frac{(17+\nu)}{96} (2f_3 - f_1)\left(\frac{r}{a}\right)^5 - \frac{(31+\nu)}{160} f_2 \left(\frac{r}{a}\right)^7 \right.\right.$$

$$\left.\left. - \frac{(49+\nu)}{240} f_3 \left(\frac{r}{a}\right)^8 + 2(1+\nu) p_2 \left(\frac{r}{a}\right) + 2(3+\nu) r_2 \left(\frac{r}{a}\right)^3 \right\}\right]$$

Thus the non-uniform portions are (i.e. terms independent of θ, at $r=a$)

$$\hat{rr}_a = \cos 2\theta \left\{ \frac{f_2}{560} - \frac{f_1}{96} + \frac{f_3}{120} - 2p_2 \right\}$$

$$\hat{\theta\theta}_a = \cos 2\theta \left\{ \frac{31}{280} f_2 - \frac{5}{32} f_1 + \frac{1}{8} f_3 + 2p_2 + 12 r_2 \right\}$$

$$\hat{r\theta}_a = \sin 2\theta \left\{ \frac{39.5}{560} f_2 - \frac{5}{96} f_1 + \frac{1}{15} f_3 + 2p_2 + 6 r_2 \right\}$$

$$\left(\frac{u}{R}\right)_a = -\cos 2\theta\left\{\frac{(13-3\nu)}{96}f_1 + \frac{(89+26\nu)}{560}f_2 + \frac{(21+5\nu)}{120}f_3 + 2(1+\nu)p_2 + 4\nu r_2\right\}$$

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$$\left(\frac{v}{R}\right)_a = \sin 2\theta\left\{-\frac{(1+\nu)}{96}f_1 + \frac{(151+15\nu)}{1120}f_2 + \frac{(9+\nu)}{60}f_3 + 2(1+\nu)p_2 + 2(3+\nu)r_2\right\}$$

$$\frac{1}{2} - 6q_2 - 4s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2\left\{\frac{f_2}{560} - \frac{f_1}{96} + \frac{f_3}{120} - 2p_2\right\}$$

$$6q_2 - \frac{1}{2} = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2\left\{\frac{31}{280}f_2 - \frac{5}{32}f_1 + \frac{1}{8}f_3 + 4p_2 + 12r_2\right\}$$

$$\frac{1}{2} + 6q_2 + 2s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2\left\{-\frac{79}{1120}f_2 + \frac{5}{96}f_1 - \frac{1}{15}f_3 - 2p_2 - 6r_2\right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4s_2 = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2\left\{-\frac{(13-3\nu)}{96}f_1 - \frac{(89+26\nu)}{560}f_2 - \frac{(21+5\nu)}{120}f_3 - 2(1+\nu)p_2 - 4\nu r_2\right\}$$

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$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2\left\{-\frac{(1+\nu)}{96}f_1 + \frac{(151+15\nu)}{1120}f_2 + \frac{(9+\nu)}{60}f_3 + 2(1+\nu)p_2 + 2(3+\nu)r_2\right\}$$

Put $\frac{E}{\sigma}\left(\frac{a}{R}\right)^2 = \eta$

3f2

$$1)\; q_2 + 0.666667 S_2 - 0.083333 = \eta \{ 0.333333 p_2 + 0 + 0.00173611 f_1 - 0.00029762 f_2 - 0.00138889 f_3 \}$$

$$2)\; q_2 + 0 - 0.083333 = \eta \{ 0.333333 p_2 + 2 r_2 - 0.02604167 f_1 + 0.01845238 f_2 + 0.02083333 f_3 \}$$

$$3)\; q_2 + 0.333333 S_2 + 0.083333 = \eta \{ -0.333333 p_2 - r_2 + 0.00868056 f_1 - 0.01175595 f_2 - 0.01111111 f_3 \}$$

$$4)\; q_2 + 1.5384615 S_2 + 0.25000 = \eta \{ - p_2 - 0.461538 r_2 - 0.04847756 f_1 - 0.06648352 f_2 - 0.07211538 f_3 \}$$

$$5)\; q_2 + 0 - 0.250000 = \eta \{ p_2 + 2.538462 r_2 - 0.00520833 f_1 + 0.05442995 f_2 + 0.05961538 f_3 \}$$

$$0.666667 S_2 + 0 = \eta \{ 0 - 2 r_2 + 0.02777778 f_1 - 0.01875000 f_2 - 0.02222222 f_3 \}$$

$$0.333333 S_2 + 0.166667 = \eta \{ -0.666667 p_2 - 3 r_2 + 0.03472222 f_1 - 0.03020833 f_2 - 0.03174444 f_3 \}$$

$$1.2051282 S_2 + 0.166667 = \eta \{ -0.666667 p_2 + 1.538462 r_2 - 0.05715812 f_1 - 0.05472757 f_2 - 0.06100427 f_3 \}$$

$$1.5384615 S_2 + 0.5000 = \eta \{ -2 p_2 - 3 r_2 - 0.04326923 f_1 - 0.12091347 f_2 - 0.13173076 f_3 \}$$

$$6)\; S_2 + 0 = \eta \{ 0 - 3 r_2 + 0.04166667 f_1 - 0.02812500 f_2 - 0.03333333 f_3 \}$$

$$7)\; S_2 + 0.50000 = \eta \{ -2 p_2 - 9 r_2 + 0.10416667 f_1 - 0.09062500 f_2 - 0.09583333 f_3 \}$$

$$8)\; S_2 + 0.1382978 = \eta \{ -0.553191 p_2 + 0.446809 r_2 - 0.04742907 f_1 - 0.0454124 f_2 - 0.05069056 f_3 \}$$

$$9)\; S_2 + 0.32500 = \eta \{ -1.3 p_2 - 1.95 r_2 - 0.02812500 f_1 - 0.07859376 f_2 - 0.08562500 f_3 \}$$

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$$0.500000 = \eta\left\{ -2p_2 - 6r_2 + 0.06250000f_1 - 0.06250000f_2 - 0.06250000f_3 \right\}$$

$$0.36170292 = \eta\left\{ -1.4468085p_2 - 9.446809r_2 + 0.15159575f_1 - 0.04521276f_2 - 0.04521277f_3 \right\}$$

$$0.18670292 = \eta\left\{ -0.7468085p_2 - 2.396809r_2 + 0.01930408f_1 - 0.03318152f_2 - 0.03500444f_3 \right\}$$

$$0.250000 = \eta\left\{ -p_2 - 3r_2 + 0.031250000f_1 - 0.03125000f_2 - 0.03125000f_3 \right\}$$

$$0.250000 = \eta\left\{ -p_2 - 6.529412r_2 + 0.10477942f_1 - 0.03125000f_2 - 0.03125000f_3 \right\}$$

$$0.250000 = \eta\left\{ -p_2 - 3.209403r_2 + 0.02584877f_1 - 0.04443110f_2 - 0.04687205f_3 \right\}$$

$$3.529412\, r_2 = 0.07352942f_1 - 0\, f_2 + 0\, f_3$$

$$3.320009\, r_2 = 0.07893065f_1 + 0.01318110f_2 + 0.01562205f_3$$

$$\boxed{r_2 = 0.02083333f_1 - 0} + 0\, f_3$$

$$r_2 = 0.02377423f_1 + 0.00397020f_2 + 0.00470542f_3$$

$$0 = 0.00294090f_1 + 0.00397020f_2 + 0.00470542f_3$$

$$\boxed{-f_3 = 0.625000f_1 + 0.843750f_2}$$

$$2\eta p_2 = -0.500000 + \eta \left\{ \begin{array}{ll} f_1 & f_2 \\ -0.1985294 & \\ +0.1360294 & -0.0625000 \\ +0.0390625 & +0.0527344 \end{array} \right\}$$

$$\boxed{\eta p_2 = -0.250000 - \eta \left\{ 0.0117188\, f_1 + 0.00488980\, f_2 \right\}}$$

$$3S_2 = \eta \left\{ \begin{array}{ll} f_1 & f_2 \\ +0.0299203 & +0.0124667 \\ -0.2406915 & \\ +0.0984043 & -0.1641622 \\ +0.1123670 & +0.1516955 \end{array} \right\}$$

$$\boxed{S_2 = -\eta \left\{ 0 + 0 \right\}}$$

$$4q_2 = \eta \left\{ \begin{array}{ll} f_1 & f_2 \\ +0.0078125 & +0.0032552 \\ +0.0112179 & \\ -0.0641026 & -0.0600648 \\ +0.0398638 & +0.0538161 \end{array} \right\}$$

$$\boxed{q_2 = -\eta \left\{ 0.0013021\, f_1 + 0.0007534\, f_2 \right\}}$$

CHECK

$$\begin{array}{ll} f_1 & f_2 \\ -0.0117188 & -0.00488980 \\ +0.0520846 & -\quad 0 \\ -0.0052083 & +0.05442995 \\ -0.0372596 & -0.05030048 \\ \hline \end{array}$$

$$r_2 = 0.02083333\, f_1 \qquad \text{O.K.}$$

$$\frac{1}{2} + r_0 = \eta\Big\{ \frac{1}{2} q_0 + \frac{f_1}{8}(1-2f_1) + \frac{1}{10} f_2(1-4f_1) + \frac{1}{24}(2f_3 - f_1 + 4f_1^2 - 8f_1 f_3 - 4.5 f_2^2)$$

$$+ \frac{6}{35} f_2(f_1 - 2f_3) + \frac{1}{96}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2) + \frac{1}{7} f_2^2$$

$$+ \frac{1}{40}(4f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2) + \frac{12}{77} f_2 f_3 + \frac{1}{240}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2) - \frac{1}{14} f_2 f_3 - \frac{1}{24} f_3^2$$

$$\frac{1}{2} - r_0 = \eta\Big\{ \frac{1}{2} q_0 + \frac{3}{8} f_1 (1-2f_1) + \cdots \Big\}$$

$$\therefore \quad q_0 = \frac{1}{\eta} - \frac{1}{2} f_1(1-2f_1) - \frac{1}{2} f_2(1-4f_1) - \frac{1}{4}(2f_3 - f_1 + 4f_1^2 - 8 f_1 f_3 - 4.5 f_2^2)$$

$$- \frac{6}{5} f_2(f_1 - 2f_3) - \frac{1}{12}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2) - \frac{2}{7} f_2^2$$

$$- \frac{1}{4}(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2) - \frac{4}{3} f_2 f_3 - \frac{1}{20}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2) + f_2 f_3 + \frac{4}{7} f_3^2$$

$$r_0 = -\eta\Big\{ \frac{1}{8} f_1(1-2f_1) + \frac{3}{20} f_2 (1-4f_1) + \frac{1}{12}(2f_3 - f_1 + 4f_1^2 - 8 f_1 f_3 - 4.5 f_2^2)$$

$$+ \frac{3}{7} f_2(f_1 - 2f_3) + \frac{1}{32}(12 f_1 f_2 + 16 f_1 f_3 - 16 f_3^2 - 3f_2 - 4f_1^2) + \frac{1}{2} f_2^2 + \frac{1}{10}(4 f_1 f_3 + 6 f_2 f_3 - f_3 - 3 f_1 f_2)$$

$$+ \frac{6}{11} f_2 f_3 + \frac{1}{48}(32 f_3^2 - 16 f_1 f_3 - 9 f_2^2) - \frac{3}{7} f_2 f_3 - \frac{1}{4} f_3^2 \Big\}$$

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Putting $f_1(1-2f_1)=A$, $f_2(1-4f_1)=B$, $(12f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 45f_2^2) = C$,

$f_2(f_1-2f_3)=D$, $(12f_1f_2+16f_1f_3-16f_3^2-3f_2-4f_1^2)=E$, $f_2^2=F$,

$(4f_1f_3+6f_2f_3-f_3-3f_1f_2)=G$, $f_2f_3=H$, $(32f_3^2-16f_1f_3-9f_2^2)=I$,

$f_2f_3=J$, $f_3^2=K$

$$\widehat{rr}+\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2\Big[\Big\{\frac{1}{7}-\frac{1}{2}A-\frac{1}{2}B-\frac{1}{4}C-\frac{6}{5}D-\frac{1}{12}E-\frac{9}{7}F-\frac{1}{4}G-\frac{4}{3}H-\frac{1}{20}I \; q_0$$

$$+J+\frac{4}{7}K\Big\}+\frac{1}{2}A\left(\frac{r}{a}\right)^2+\frac{1}{2}B\left(\frac{r}{a}\right)^3+\frac{1}{4}C\left(\frac{r}{a}\right)^4+\frac{6}{5}D\left(\frac{r}{a}\right)^5+\frac{1}{12}E\left(\frac{r}{a}\right)^6+\frac{9}{7}F\left(\frac{r}{a}\right)^7$$

$$+\frac{1}{4}G\left(\frac{r}{a}\right)^8+\frac{4}{3}H\left(\frac{r}{a}\right)^9+\frac{1}{20}I\left(\frac{r}{a}\right)^{10}-J\left(\frac{r}{a}\right)^{12}-\frac{4}{7}K\left(\frac{r}{a}\right)^{14}+$$

$$\cos 2\theta\Big\{\frac{3}{10}f_2\left(\frac{r}{a}\right)^3+\frac{1}{6}(2f_3-f_1)\left(\frac{r}{a}\right)^4-\frac{3}{16}f_2\left(\frac{r}{a}\right)^6-\frac{1}{5}f_3\left(\frac{r}{a}\right)^8+0 \quad +12r_2\left(\frac{r}{a}\right)^2\Big\}\Big]$$

$$\int_0^{2\pi}\left(\widehat{rr}+\widehat{\theta\theta}\right)^2 d\theta = \pi E^2\left(\frac{a}{R}\right)^4\Big[2\Big\{q_0^2+Aq_0\left(\frac{r}{a}\right)^2+Bq_0\left(\frac{r}{a}\right)^3+\left(\frac{A^2}{4}+\frac{1}{2}Cq_0\right)\left(\frac{r}{a}\right)^4+\left(\frac{1}{2}AB+\frac{12}{5}Dq_0\right)\left(\frac{r}{a}\right)^5$$

$$+\left(\frac{1}{4}B^2+\frac{1}{6}Eq_0\right)\left(\frac{r}{a}\right)^6+\left(\frac{1}{4}BC+\frac{18}{7}Fq_0\right)\left(\frac{r}{a}\right)^7+\left(\frac{1}{16}C^2+\frac{1}{2}Gq_0\right)\left(\frac{r}{a}\right)^8+($$

$$\int_0^{2\pi} (\dot{n}\dot{n}+\dot{\theta}\dot{\theta})^2 \, d\varphi = \pi E^2 \left(\frac{a}{R}\right)^4 \Big[2\Big\{ q_0^2 + A q_0 \left(\frac{r}{a}\right)^2 + B q_0 \left(\frac{r}{a}\right)^3 + \left(\frac{A^2}{4} + \frac{1}{2} C q_0\right)\left(\frac{r}{a}\right)^4 + \left(\frac{1}{2}AB + \frac{12}{5} D q_0\right)\left(\frac{r}{a}\right)^5$$

$$+\left(\frac{1}{4}B^2 + \frac{1}{4}AC + \frac{1}{6}E q_0\right)\left(\frac{r}{a}\right)^6 + \left(\frac{1}{4}BC + \frac{6}{5}AD + \frac{18}{7}F q_0\right)\left(\frac{r}{a}\right)^7 + \left(\frac{1}{16}C^2 + \frac{6}{5}BD + \frac{1}{12}AE + \frac{1}{2}G q_0\right)\left(\frac{r}{a}\right)^8$$

$$+\left(\frac{3}{5}CD + \frac{1}{12}BE + \frac{9}{7}AF + \frac{1}{3}H q_0\right)\left(\frac{r}{a}\right)^9 + \left(\frac{36}{25}D^2 + \frac{1}{24}CE + \frac{9}{7}BF + \frac{1}{4}AG + \frac{1}{10}I q_0\right)\left(\frac{r}{a}\right)^{10}$$

$$+\left(\frac{1}{5}DE + \frac{9}{14}CF + \frac{1}{4}BG + \frac{4}{3}AH\right)\left(\frac{r}{a}\right)^{11} +$$

$$+\left(\frac{1}{144}E^2 + \frac{108}{35}DF + \frac{1}{8}CG + \frac{4}{3}BH + \frac{1}{40}AI - 2J q_0\right)\left(\frac{r}{a}\right)^{12} + \left(\frac{3}{14}EF + \frac{3}{5}DG + \frac{2}{3}CH\right.$$

$$\left. + \frac{1}{20}BI\right)\left(\frac{r}{a}\right)^{13} + \left(\frac{81}{49}F^2 + \frac{1}{24}EG + \frac{48}{15}DH + \frac{1}{40}CI - AJ - \frac{8}{7}K q_0\right)\left(\frac{r}{a}\right)^{14}$$

$$+\left(\frac{9}{14}FG + \frac{4}{18}EH + \frac{6}{50}DI - BJ\right)\left(\frac{r}{a}\right)^{15} +$$

$$+\left(\frac{1}{16}G^2 + \frac{24}{7}FH + \frac{1}{120}EI - \frac{1}{2}CJ - \frac{4}{7}AK\right)\left(\frac{r}{a}\right)^{16} + \left(\frac{2}{3}GH + \frac{9}{70}FI - \frac{12}{5}DJ - \frac{4}{7}BK\right)\left(\frac{r}{a}\right)^{17}$$

$$+\left(\frac{16}{9}H^2 + \frac{1}{40}GI - \frac{1}{6}EJ - \frac{2}{7}CK\right)\left(\frac{r}{a}\right)^{18} + \left(\frac{4}{30}HI - \frac{18}{7}FJ - \frac{48}{35}DK\right)\left(\frac{r}{a}\right)^{19}$$

$$+\left(\frac{1}{400}I^2 - \frac{1}{2}GJ - \frac{2}{21}EK\right)\left(\frac{r}{a}\right)^{20} + \left(-\frac{8}{3}HJ - \frac{72}{49}FK\right)\left(\frac{r}{a}\right)^{21} + \left(-\frac{1}{10}IJ - \frac{2}{7}GK\right)\left(\frac{r}{a}\right)^{22}$$

$$+\left(J^2 - \frac{4}{70}IK\right)\left(\frac{r}{a}\right)^{24} + \frac{8}{7}JK\left(\frac{r}{a}\right)^{26} + \frac{16}{49}K^2\left(\frac{r}{a}\right)^{28} - \frac{32}{21}HK\left(\frac{r}{a}\right)^{23}$$

$$+\Big\{\frac{9}{120}f_2^2\left(\frac{r}{a}\right)^6 + \frac{1}{10}f_2(2f_3 - f_1)\left(\frac{r}{a}\right)^7 + \left[\frac{1}{36}(2f_3 - f_1)^2 - \frac{9}{2}f_2 h_2\right]\left(\frac{r}{a}\right)^8$$

3f

$$+\Big\{144 h_3^2\left(\tfrac{r}{a}\right)^4+\frac{36}{5}f_2 h_2\left(\tfrac{r}{a}\right)^5+\Big[\frac{9}{100}f_2^2+4(2f_3-f_1)h_2\Big]\left(\tfrac{r}{a}\right)^6+\frac{1}{10}f_2(2f_3-f_1)\ldots\left(\tfrac{r}{a}\right)^7$$

$$+\Big[\frac{1}{36}(2f_3-f_1)^2-\frac{9}{2}f_3 h_2\Big]\left(\tfrac{r}{a}\right)^8-\frac{9}{80}f_2^2\left(\tfrac{r}{a}\right)^9-\Big[\frac{1}{16}f_2(2f_3-f_1)+\frac{24}{5}f_3 h_3\Big]\left(\tfrac{r}{a}\right)^{10}$$

$$-\frac{6}{50}f_2 f_3\left(\tfrac{r}{a}\right)^{11}+\Big[\frac{9}{25}f_2^2-\frac{1}{15}f_3(2f_3-f_1)\Big]\left(\tfrac{r}{a}\right)^{12}+\frac{3}{40}f_2 f_3\left(\tfrac{r}{a}\right)^{14}+\frac{1}{25}f_3^2\left(\tfrac{r}{a}\right)^{16}\Big\}$$

$$\int_0^{2\pi}\hat{rr}\cdot\hat{\theta\theta}\,d\theta=\pi E^2\left(\tfrac{a}{R}\right)^4\Big[2\Big\{\frac{1}{4}q_0^2+\frac{1}{4}Aq_0\left(\tfrac{r}{a}\right)^2+\frac{1}{4}Bq_0\left(\tfrac{r}{a}\right)^3+\left(\frac{3}{64}A^2+\frac{1}{8}Cq_0\right)\left(\tfrac{r}{a}\right)^4$$

$$+\left(\frac{7}{80}AB+\frac{3}{5}Dq_0\right)\left(\tfrac{r}{a}\right)^5+\left(\frac{4}{100}B^2+\frac{1}{24}AC+\frac{1}{24}Eq_0\right)\left(\tfrac{r}{a}\right)^6+\left(\frac{3}{80}BC+\frac{27}{140}AD+\frac{9}{14}Fq_0\right)\left(\tfrac{r}{a}\right)^7$$

$$+\left(\frac{5}{24^2}C^2+\frac{6}{35}BD+\frac{5}{48\times8}AE+\frac{1}{8}Gq_0\right)\left(\tfrac{r}{a}\right)^8+\left(\frac{11}{140}CD+\frac{11}{960}BE+\frac{11}{56}AF+\frac{12}{18}Hq_0\right)\left(\tfrac{r}{a}\right)^9$$

$$+\left(\frac{9.16}{35^2}D^2+\frac{1}{192}CE+\frac{12}{70}BF+\frac{3}{80}AG+\frac{1}{40}Iq_0\right)\left(\tfrac{r}{a}\right)^{10}+\left(\frac{13}{35\times16}DE+\frac{13}{24\times7}CF+\frac{13}{400}BG+\frac{13}{66}AH\right)\left(\tfrac{r}{a}\right)^{11}$$

$$+\left(\frac{7}{96}E^2+\frac{12}{35}DF+\frac{7}{480}CG+\frac{24}{165}BH+\frac{7}{960}AI-\frac{1}{2}Jq_0\right)\left(\tfrac{r}{a}\right)^{12}+$$

$$+\left(\frac{5}{33\times7}EF+\frac{9}{140}DG+\frac{5}{66}CH+\frac{1}{160}BI\right)\left(\tfrac{r}{a}\right)^{13}+\left(\frac{1}{49}F^2+\frac{1}{240}EG+\frac{134}{385}DH+\frac{1}{360}CI-\frac{1}{7}AJ-\frac{2}{7}Kq_0\right)\left(\tfrac{r}{a}\right)^{14}$$

$$+\left(\frac{17}{980}FG+\frac{17}{33\times24}EH+\frac{17}{1400}DI-\frac{17}{140}BJ\right)\left(\tfrac{r}{a}\right)^{15}+$$

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$$+\left(\frac{9}{1600}G^2+\frac{24}{77}FH+\frac{1}{1280}EI-\frac{3}{56}CJ-\frac{9}{112}AK\right)\left(\frac{r}{a}\right)^{16}$$

$$+\left(\frac{19}{330}GH+\frac{19}{7\times 140}FI-\frac{57}{245}DJ-\frac{19}{280}BK\right)\left(\frac{r}{a}\right)^{17}$$

$$+\left(\frac{160}{1089}H^2+\frac{1}{480}GI-\frac{5}{24\times 14}EJ-\frac{5}{24\times 7}CK\right)\left(\frac{r}{a}\right)^{18}$$

$$+\left(\frac{7}{660}HI-\frac{3}{14}FJ-\frac{9}{70}DK\right)\left(\frac{r}{a}\right)^{19}+\left(\frac{11}{2\cdot 480}I^2-\frac{11}{480}GJ-\frac{11}{14\times 16}EK\right)\left(\frac{r}{a}\right)^{20}$$

$\frac{11}{280}$

$$+\left(-\frac{46}{231}HJ-\frac{23}{28\times 7}FK\right)\left(\frac{r}{a}\right)^{21}+\left(-\frac{1}{140}IJ-\frac{3}{140}GK\right)\left(\frac{r}{a}\right)^{22}$$

$$+\left(\frac{13}{14^2}J^2-\frac{13}{120\times 28}IK\right)\left(\frac{r}{a}\right)^{24}+\frac{1}{14}JK\left(\frac{r}{a}\right)^{26}+\frac{15}{28^2}K^2\left(\frac{r}{a}\right)^{28}-\frac{25}{231}HK\left(\frac{r}{a}\right)^{23}$$

$$+\Big\{-4p_2^2-24p_2r_2\left(\frac{r}{a}\right)^2-\frac{19}{35}f_2p_2\left(\frac{r}{a}\right)^3-\frac{7}{24}(2f_3-f_1)p_2\left(\frac{r}{a}\right)^4+\left[\frac{13}{40}f_3 r_2+\frac{20}{4900}f_2^2+\frac{1}{8}(2f_3-f_1)r_2\right]\left(\frac{r}{a}\right)^6$$

$$+\frac{16}{35}f_3 r_2\left(\frac{r}{a}\right)^5$$

$$+\frac{1}{192}f_2(2f_3-f_1)\left(\frac{r}{a}\right)^7+\left[\frac{15}{16^2}(2f_3-f_1)^2-\frac{3}{20}f_2r_2+\frac{7}{90}f_3p_2\right]\left(\frac{r}{a}\right)^8$$

$$-\frac{17}{2800}f_2^2\left(\frac{r}{a}\right)^9-\left[\frac{29}{96\times 80}f_2(2f_3-f_1)+\frac{3}{20}f_3r_2\right]\left(\frac{r}{a}\right)^{10}-\frac{1}{160}f_2f_3\left(\frac{r}{a}\right)^{11}$$

$$+\left[\frac{14}{6400}f_2^2-\frac{1}{8\times 32}f_3(2f_3-f_1)\right]\left(\frac{r}{a}\right)^{12}+\frac{27}{6400}f_2f_3\left(\frac{r}{a}\right)^{14}+\frac{15}{6400}f_3^2\left(\frac{r}{a}\right)^{16}\Big\}$$

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$$\int_0^{2\pi} \widehat{r\theta}^2 \, d\theta = E^2\left(\frac{a}{R}\right)^4 \pi \Big[4p_2^2 + 24 p_2 r_2 \left(\frac{r}{a}\right)^2 + \frac{16}{35} f_2 p_2 \left(\frac{r}{a}\right)^3 + \left\{36 r_2^2 + \frac{5}{24}(2f_3 - f_1) p_2\right\}\left(\frac{r}{a}\right)^4$$

$$+ \frac{48}{35} f_2 r_2 \left(\frac{r}{a}\right)^5 + \left\{\frac{64}{4900} f_2^2 + \frac{5}{8}(2f_3-f_1) r_2\right\}\left(\frac{r}{a}\right)^6 + \frac{5}{420} f_2(2f_3-f_1)\left(\frac{r}{a}\right)^7$$

$$+ \left\{\frac{25}{96^2}(2f_3-f_1)^2 - \frac{10.5}{20} f_2 r_2 - \frac{3}{20} f_3 p_2\right\}\left(\frac{r}{a}\right)^8 - \frac{1}{100} f_2^2 \left(\frac{r}{a}\right)^9 - \left\{\frac{7}{96\times 16} f_2(2f_3-f_1)\right.$$

$$\left. + \frac{9}{20} f_3 r_2\right\}\left(\frac{r}{a}\right)^{10} - \frac{3}{350} f_2 f_3 \left(\frac{r}{a}\right)^{11} + \left[\frac{3.5^2}{80^2} f_2^2 - \frac{3}{8\times 96} f_3(2f_3-f_1)\right]\left(\frac{r}{a}\right)^{12}$$

$$+ \frac{21}{80^2} f_2 f_3 \left(\frac{r}{a}\right)^{14} + \frac{9}{6400} f_3^2 \left(\frac{r}{a}\right)^{16} \Big]$$

The part from non-uniform part

$$\int_0^{2\pi} \left[\widehat{rr}\cdot\widehat{\theta\theta} - \widehat{r\theta}^2\right] d\theta = E^2\left(\frac{a}{R}\right)^4 \pi \Big[-8p_2^2 - 48 p_2 r_2 \left(\frac{r}{a}\right)^2 - f_2 p_2 \left(\frac{r}{a}\right)^3 - \left\{36 r_2^2 + \frac{1}{2}(2f_3-f_1) p_2\right\}\left(\frac{r}{a}\right)^4$$

$$- \frac{6}{5} f_2 r_2 \left(\frac{r}{a}\right)^5 - \left\{\frac{44}{4900} f_2^2 - \frac{1}{2} f_2 + \frac{1}{2}(2f_3-f_1) r_2\right\}\left(\frac{r}{a}\right)^6 \; \frac{3}{448} f_2(2f_3-f_1)\left(\frac{r}{a}\right)^7$$

$$\frac{1}{1280}$$

$$- \left\{\frac{10}{96^2}(2f_3-f_1)^2 - \frac{7.5}{20} f_2 r_2 - \frac{1}{2} f_3 p_2\right\}\left(\frac{r}{a}\right)^8 + \frac{11}{9800} f_2^2 \left(\frac{r}{a}\right)^9 + \left\{\frac{1}{12800} f_2 (2f_3 - f_1)\right.$$

$$\left. + \frac{3}{10} f_3 r_2\right\}\left(\frac{r}{a}\right)^{10} + \frac{13}{350\times 16} f_2 f_3 \left(\frac{r}{a}\right)^{11} + \frac{1.75}{6400} f_2^2 \left(\frac{r}{a}\right)^{12} + \frac{1}{800} f_2 f_3 \left(\frac{r}{a}\right)^{14} + \frac{3}{3200} f_3^2 \left(\frac{r}{a}\right)^{16}$$

$$\frac{\mathcal{C}_1 / R^3}{\frac{\pi E}{2}\left(\frac{A}{R}\right)^6\left(\frac{t}{R}\right)} = 0.35\, q_0^2 + 0.175\, A q_0 + 0.140\, B q_0 + (0.04270833\, A^2 + 0.0583333\, C q_0) + (0.0778571\, AB$$

$$+ 0.24000\, D q_0) + (0.0365000\, B^2 + 0.03541667\, AC + 0.0145833\, E q_0) + (0.0338889\, BC + 0.1552381\, AD$$

$$+ 0.20000\, F q_0) + (\cancel{0.00798611}\, C^2 + 0.15085743\, BD + 0.00989583\, AE + 0.036000\, G q_0) +$$

$$+ (\cancel{0.07194805\, CD} + 0.009734848\, BE + 0.14090909\, AF + 0.16969696\, H q_0) + (\cancel{0.1635718\, D}^2 +$$

$$+ \cancel{0.0046875\, CE} + 0.1400000\, BF + 0.03541667\, AG + 0.005833333\, I q_0) + (\cancel{0.02148351\, DE} +$$

$$+ \cancel{0.067948718\, CF} + 0.0254615\, BG + 0.1263403\, AH) + (\cancel{0.00070995\, E^2} + \cancel{0.3134694\, DF}$$

$$+ \cancel{0.0727405}\, CG + 0.1274459\, BH + 0.00443452\, AI - 0.100000\, J q_0) + (\cancel{0.02083333\, EF} +$$

$$+ \cancel{0.0517143\, DG} + 0.0626262\, CH + 0.0045000\, BI) + (0.1535714\, F^2 + \cancel{0.00385411\, EG} +$$

$$+ 0.\cancel{2117481}\, DH + 0.00\cancel{922222\, CI} - 0.07857143\, AJ - 0.050000\, K q_0) + (0.0570588\, FG + \cancel{0.0119781\, EH}$$

$$+ \cancel{0.0104033}\, DI - 0.0805042\, BJ) + (\cancel{0.0053444}\, G^2 + \cancel{0.2909090\, FH} + \cancel{0.000700\, EI}$$

$$0.0104\cancel{3}359651$$

$$- \cancel{0.04007137}\, CJ - 0.04027777\, AK) + (\cancel{0.0544176}\, GH + 0.0100626\, FI - \cancel{0.1889581\, DJ} - \cancel{0.04157898}\, K)$$

$$+ (\cancel{0.1395776\, H}^2 + 0.00\cancel{125833}\, GI - \cancel{0.0127762\, EJ} - 0.0\cancel{20}83333\, CK)$$

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$$+\left(0.01007215\,HI - 0.1918367\,FJ - 0.00987755\,DK\right) + \left(0.0001821338\,I^2 - 0.04003788\,GI\right.$$

$$0.0361688511$$

$$\left.-\,0.00672348\,EK\right) - 0.1868624\,HJ - \overset{1012422\;6}{0.10133097}\,FK - 0.006785714\,IJ - 0.01916667\,GK$$

$$+\,0.06365777\,J^2 - 0.003621779\,IK + 0.06836735\,JK + 0.01845238\,K^2 - 0.0993939\overset{44}{39}\,HK$$

$$+\Big\{10.4\,p_2^2 + 31.2\,p_2 r_2 + 0.52\,f_3 p_2 + 39.6\,r_2^2 + 0.21666667\,(2f_3 - f_1)\,p_2 + 1.4742857\,f_2 r_2$$

$$+\,0.01416837\,f_2^2 - 0.162500\,f_2 p_2 + 0.662500\,(2f_3 - f_1)\,r_2 + 0.01304563\,f_2\,(2f_3 - f_1)$$

$$+\,0.003059896\,(2f_3 - f_1)^2 - 0.547500\,f_2 r_2 - 0.13000\,f_3 p_2 - 0.01115844\,f_2^2 - \overset{0.005377604}{0.00522526\,f_2\,(2f_3 - f_1)}$$

$$-\,0.465\,f_3 r_2 - 0.00969505\,f_2 f_3 + 0.002460379\,f_2^2 - 0.004761905\,f_3\,(2f_3 - f_1) + 0.004484375\,f_2 f_3$$

$$+\,0.002086806\,f_3^2\Big\}$$

$$2f_3 - f_1 + 4f_1^2 - 8f_1f_3 - 4.5f_2^2$$

$$= -2.25000f_1 - 1.687500f_2 + 9f_1^2 + 6.750000f_1f_2 - 4.5000f_2^2$$

$$f_2(f_1 - 2f_3) = 2.250000f_1f_2 + 1.6875000f_2^2$$

$$12f_1f_2 + 16f_1f_3 - 16f_3^2 - 3f_2 - 4f_1^2$$

$$= -3f_2 - 20.25000f_1^2 - 18.37500f_1f_2 - 11.3906250f_2^2$$

$$4f_1f_3 + 6f_2f_3 - f_3 - 3f_1f_2 = 0.6250f_1 + 0.84375f_2 - 2.5000f_1^2 - 10.12500f_1f_2 - 5.062500f_2^2$$

$$f_2f_3 = -0.625000f_1f_2 - 0.84375f_2^2$$

$$32f_3^2 - 16f_1f_3 - 9f_2^2 = 16f_3(2f_3 - f_1) - 9f_2^2$$

$$= 16(0.625f_1 + 0.84375f_2)(2.2500f_1 + 1.68750f_2) - 9f_2^2$$

$$= 22.500f_1^2 + 47.2500f_1f_2 + 13.781250f_2^2$$

$$f_3^2 = 0.390625f_1^2 + 1.0546875f_1f_2 + 0.7119140625f_2^2$$

$$f_1(1 - 2f_1) = f_1 - 2f_1^2$$

$$f_2(1 - 4f_1) = f_2 - 4f_1f_2$$

To calculate the bending energy: 3R5

$$k_1 = \frac{\partial^2 w}{\partial r^2} - \frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\left\{ 2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 3f_2\left[2-5\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[3-7\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}$$

$$k_2 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}$$

$$= \frac{1}{R}\left\{ 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] + 3f_2\left[1-\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 4f_3\left[1-\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}$$

$$k_1+k_2 = \frac{1}{R}\left\{ 4f_1\left[1-2\left(\frac{r}{a}\right)^2\right] + 9f_2\left[1-2\left(\frac{r}{a}\right)^3\right]\left(\frac{r}{a}\right) + 16f_3\left[1-2\left(\frac{r}{a}\right)^4\right]\left(\frac{r}{a}\right)^2\right\}$$

$$= \frac{1}{R}\left\{ 4f_1 + 9f_2\left(\frac{r}{a}\right) + 8(2f_3-f_1)\left(\frac{r}{a}\right)^2 - 18f_2\left(\frac{r}{a}\right)^4 - 32f_3\left(\frac{r}{a}\right)^6\right\}$$

$$\int_0^a (k_1+k_2)^2 r\,dr = \left(\frac{a}{R}\right)^2\left\{ 8f_1^2 + 24f_1f_2 + \frac{81}{4}f_2^2 + 16f_1(2f_3-f_1)\right.$$

$$+ \frac{144}{5}f_2(2f_3-f_1) + \frac{32}{3}(2f_3-f_1)^2 - 24f_1f_2 - \frac{324}{7}f_2^2 - 32f_1f_3$$

$$- 36f_2(2f_3-f_1) - 64f_2f_3 + 32.4f_2^2 - 51.2f_3(2f_3-f_1) + 96f_2f_3$$

$$\left. + \frac{512}{7}f_3^2\right\}$$

$$\int_0^a (k_1k_2)r\,dr = \left(\frac{a}{R}\right)^2\left\{ 2f_1^2 + 6f_1f_2 + \frac{9}{2}f_2^2 + 4f_1(2f_3-f_1)\right.$$

$$+6f_2(2f_3-f_1) + 2(2f_3-f_1)^2 - 6f_2f_1 - 9f_2^2 - 6f_2(2f_3-f_1)$$

$$\left. -8f_1f_3 - 12f_2f_3 + 4.5f_2^2 - 8f_3(2f_3-f_1) + 12f_2f_3 + 8f_3^2\right\}$$

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$$\int_0^a (k_1+k_2)^2 r\,dr = \left(\frac{a}{R}\right)^2 \Big\{ 8f_1^2 + 0 + 6.364286 f_2^2 + 16 f_1(2f_3 - f_1)$$

$$- 7.2 f_2(2f_3 - f_1) + \frac{32}{3}(2f_3 - f_1)^2 - 32 f_1 f_3 + 32 f_2 f_3 - 51.2 f_3 (2f_3 - f_1)$$

$$+ \frac{512}{7} f_3^2 \Big\}$$

$$\int_0^a (k_1 k_2) r\,dr = \left(\frac{a}{R}\right)^2 \Big\{ 2f_1^2 + 0 + 0 + 4f_1(2f_3 - f_1) + 0$$

$$+ 2(2f_3 - f_1)^2 - 8 f_1 f_3 + 0 - 8 f_3 (2f_3 - f_1) + 8 f_3^2 \Big\}$$

$$\frac{\mathcal{E}_2}{R^3} = \frac{1}{12}\left(\frac{t}{R}\right)^3 \frac{E\pi}{(1-\nu^2)}\left(\frac{a}{R}\right)^2 \Big\{ 2.8 f_1^2 + 6.364286 f_2^2 + 5.6(2f_3 - f_1) f_1$$

$$- 7.2 f_2 (2f_3 - f_1) + 5.4666667 (2f_3 - f_1)^2 - 11.2 f_1 f_3 + 32 f_2 f_3$$

$$- 30.4 f_3 (2f_3 - f_1) + 52.34285714 f_3^2 \Big\}$$

$$\frac{\dfrac{\mathcal{E}_2}{R^3}}{\dfrac{\pi E}{2}\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)^3} = 0.5128205 f_1^2 + 1.1656201 f_2^2 + 1.0256410 f_1(2f_3 - f_1)$$

$$- 1.3186813 f_2 (2f_3 - f_1) + 1.0012210 (2f_3 - f_1)^2$$

$$- 2.0512421 f_1 f_3 + 5.8608059 f_2 f_3 - 5.5677656 f_3 (2f_3 - f_1)$$

$$+ 9.5866039 f_3^2$$

$$\frac{\mathcal{E}_3/R^3}{\frac{1}{2}\frac{\sigma^2}{E}\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2} = 8s_2^2 + 2(1+\nu)\left\{(r_0^2+2s_2^2)+12\,s_2q_2+12q_2^2\right\}$$

$$\frac{\mathcal{K}\theta/R^3}{\frac{\sigma^2}{2E}\pi\left(\frac{t}{R}\right)\left(\frac{a}{R}\right)^2} = 4s_2 - 2(1+\nu)\,r_0$$

For the region outside the circle:

$$\widehat{rr} = \sigma\left[\frac{1}{2} + \frac{r_0}{\left(\frac{r}{a}\right)^2} + \cos 2\theta\left\{\frac{1}{2} - \frac{6q_2}{\left(\frac{r}{a}\right)^4} - \frac{4s_2}{\left(\frac{r}{a}\right)^2}\right\}\right]$$

$$\widehat{\theta\theta} = \sigma\left[\frac{1}{2} - \frac{r_0}{\left(\frac{r}{a}\right)^2} + \cos 2\theta\left\{\frac{6q_2}{\left(\frac{r}{a}\right)^4} - \frac{1}{2}\right\}\right]$$

$$\widehat{r\theta} = -\sigma\sin 2\theta\left\{\frac{1}{2} + \frac{6q_2}{\left(\frac{r}{a}\right)^4} + \frac{2s_2}{\left(\frac{r}{a}\right)^2}\right\}$$

$$\frac{u}{R} = \frac{\sigma}{E}\left(\frac{a}{R}\right)\left[\frac{1}{2}(1-\nu)\left(\frac{r}{a}\right) - (1+\nu)\frac{r_0}{\left(\frac{r}{a}\right)} + \cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{r}{a}\right) + 2(1+\nu)\frac{q_2}{\left(\frac{r}{a}\right)^3} + \frac{4s_2}{\left(\frac{r}{a}\right)}\right\}\right]$$

$$\frac{v}{R} = \frac{\sigma}{E}\left(\frac{a}{R}\right)\left[2(1+\nu)\frac{q_2}{\left(\frac{r}{a}\right)^3} - \frac{1}{2}(1+\nu)\left(\frac{r}{a}\right)\right]\sin 2\theta$$

$$\frac{\mathscr{E}_2/R^3}{\frac{\pi E}{2}\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)^3} = 0.5128205\,f_1^2 + 1.1656201\,f_2^2 - 1.0256410\,f_1\,(2.2500 f_1 + 1.6875 f_2)$$

$$+ 1.3186813\,f_2\,(2.2500 f_1 + 1.6875 f_2) + 1.0012210\,(2.2500 f_1 + 1.6875 f_2)^2$$

$$+ 2.0512821\,f_1\,(0.625 f_1 + 0.84375 f_2) - 5.8608059\,f_2\,(0.625 f_1 + 0.84375 f_2)$$

$$- 5.5677156\,(0.625 f_1 + 0.84375 f_2)(2.250 f_1 + 1.6875 f_2) + 9.5866039\,(0.625 f_1 + 0.84375 f_2)^2$$

=

f_1^2	$f_1 f_2$	f_2^2
+0.5128205		+1.1656201
−2.3076923	−1.7307692	
	+2.9670329	+2.2252747
+5.0686813	+7.6030220	+2.8511332
+1.2820513	+1.7307693	
	−3.6630037	−4.9450550
−7.8296704	−16.4423078	−7.9275413
+3.7447671	+10.1108713	+6.8248381
+0.4709575	+0.5756148	+0.1942698

$$\mathscr{E}_2/R^3 = \frac{\pi E}{2}\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)^3\left\{0.4709575\,f_1^2 + 0.5756148\,f_1 f_2 + 0.1942698\,f_2^2\right\}$$

$q_0 - \frac{1}{7} =$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
−0.500000		+1.00000		
	−0.50000		+2.00000	
+0.562500	+0.421875	−2.25000	−1.6875	+1.125000
			−2.7000	−2.025000
	+0.750000	+1.6875	+1.53125	+0.94921875
				−1.28571429
−0.15625	−0.2109375	+0.625	+2.53125	+1.265625
			+0.2083333	+0.2812500
		−1.125	−2.3625	−0.6890625
		+0.22321429	+0.60267857	+0.40680804

$-0.093750 f_1 - 0.03906250 f_2 + 0.16071429 f_1^2 + 0.12351190 f_1 f_2 + 0.02812500 f_2^2$

0.12351183

$r_0 =$ f_1	f_2	f_1^2	$f_1 f_2$	f_2^2	<u>400</u>
−0.1250		+0.2500			
	−0.150		+0.60000		
+0.18750	+0.140625	−0.25000	−0.562500	+0.37500	
			−0.96428571	−0.72321429	
	+0.09375	+0.632[illegible]125	+0.5742[illegible]625	+0.35595[illegible]	
				−[illegible].5000	
−0.0625	−0.084375	+0.25	+1.0125	+0.50625	
			+0.07305195	+0.0986[illegible]13	
		−0.46[illegible]25	−0.984375	−0.2[illegible]	
		+0.09765625	+0.26367[illegible]625	+0.12798852	

$$\eta\{0 + 0 + 0.01[illegible]\,f_1^2 + 0.0[illegible]\,f_1 f_2 + 0.00[illegible]\,f_2^2\}$$

$$r_0^2 = \eta^2\{0.0001373291\,f_1^4 + 0.0002878563\,f_1^3 f_2 + 0.0002324539\,f_1^2 f_2^2 + 0.0000855334\,f_1 f_2^3 + 0.0000121244\,f_2^4\}$$

$$2\eta_2^2 = \eta^2\{0.0000203455\,f_1^2 + 0.0000235440\,f_1 f_2 + 0.0000068113\,f_2^2\}$$

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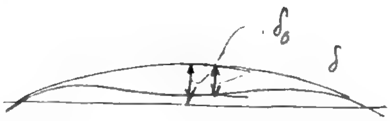
$$\mathcal{E}_3 / R^3 = \frac{1}{2}\frac{\sigma^2}{E}\pi\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)\Big\{ 0.0003570557 f_1^4 + 0.0007484264 f_1^3 f_2$$

$$+ 0.0006043801 f_1^2 f_2^2 + 0.0002223868 f_1 f_2^3 + 0.0000315234 f_2^4$$

$$+ 0.0000528983 f_1^2 + 0.0000612144 f_1 f_2 + 0.0000177094 f_2^2 \Big\} \eta^2$$

$$\delta\Omega / R^3 = \frac{1}{2}\frac{\sigma^2}{E}\pi\left(\frac{a}{R}\right)^2\left(\frac{t}{R}\right)\eta \Big\{ -0.03046875 f_1^2 - 0.03193286 f_1 f_2 - 0.00905323 f_2^2 \Big\}$$



$$\frac{\delta}{\delta_0} = f_1 + f_2 + f_3$$

$$= 0.375 f_1 + 0.15625 f_2$$

The terms in { } in p. 393 can be collected as

$$\eta_1 \Big\{ 10.4 f_2^2 + 31.2 p_2 r_2 + 0.3525 f_2 p_2 + 39.6 r_2^2 - 0.21666667 f_2 (2.250 f_1 + 1.6875 f_2)$$

$$+ 0.9267857 f_2 r_2 + 0.00547291 f_2^2 - 0.6625000 (2.2500 f_1 + 1.6875 f_2) r_2$$

$$- 0.00766803 f_2 (2.2500 f_1 + 1.6875 f_2) + 0.003059896 (2.2500 f_1 + 1.6875 f_2)^2$$

$$+ 0.13000 p_2 (0.625 f_1 + 0.84375 f_2) + 0.465 r_2 (0.625 f_1 + 0.84375 f_2)$$

$$+ 0.0052067 f_2 (0.625 f_1 + 0.84375 f_2) - 0.004761905 (0.625 f_1 + 0.84375 f_2)$$

$$(2.25 f_1 + 1.6875 f_2) + 0.002086806 (0.625 f_1 + 0.84375 f_2)^2 \Big\}$$

$$= 0.6500000 +$$

$$+\eta\Big\{\big(\overset{0.0609378}{0.0058594\,f_1} + \overset{0.0253906}{0.00244140\,f_2}\big)\times 10.4 - 0.1625000\,f_1 - 0.089375\,f_2$$

$$+ 0.121875\,f_1 + 0.0914062\,f_2 - 0.0203125\,f_1 - 0.02742187\,f_2\Big\}$$

$$0 + \; 0$$

$$+\eta^2\Big\{$$

f_1	f_2	f_1^2	f_1f_2	f_2^2
		+0.00142824	+0.00119019	+0.00024795
		−0.00761722	−0.00317382	
			−0.00418947	−0.00174560 −0.00160260
		+0.0171875		
			+0.01930804	
				+0.00542271
		−0.03105469	−0.02329102	
			−0.01725307	−0.01293980
		+0.01549072	+0.02333609	+0.00871353
		+0.00605469	+0.00817383	
		−0.0011205 −0.00683343	+0.00325667 −0.00687865 −0.01351121	+0.00439650 −0.00404481 −0.00578621
		$-0.00534419\,f_1^2$	$-0.00625377\,f_1f_2$	$-0.00149772\,f_2^2$
		+0.00036874	+0.0037879	+0.00010068

Independent check

η^2	f_1^2	$f_1 f_2$	f_2^2
	+0.00142824	+0.00119019	+0.00027295
	−0.00761722	−0.00317582	
		−0.00418947	−0.00174560
	+0.01718725		
		+0.01930804	
			+0.00547291
	−0.00985104	−0.01064295	−0.00144103
	−0.00077874	−0.00211321	−0.00143358
	+0.00036874	+0.00037828	+0.00010068

$$A = f_1 - 2f_1^2$$

$$B = f_2 - 4f_1f_2$$

$$C = -2.25000f_1 - 1.6875f_2 + 9f_1^2 + 6.750000f_1f_2 - 4.50000f_2^2$$

$$D = 2.25000f_1f_2 + 1.687500f_2^2$$

$$E = -3f_2 - 20.25000f_1^2 - 18.37500f_1f_2 - 11.3906250f_2^2$$

$$F = f_2^2$$

$$G = 0.625f_1 + 0.84375f_2 - 2.5000f_1^2 - 10.12500f_1f_2 - 5.0625000f_2^2$$

$$H = -0.6250f_1f_2 - 0.84375f_2^2 = J$$

$$I = 22.500f_1^2 + 47.2500f_1f_2 + 13.78125f_2^2$$

$$K = 0.390625f_1^2 + 1.0546875f_1f_2 + 0.7119140625f_2^2$$

$$\eta q_0 = 1 + \eta\Big\{-0.09375f_1 - 0.0390625f_2 + 0.16071429f_1^2 + 0.12351190f_1f_2 + 0.07812500f_2^2\Big\}$$

$$\eta^2 q_0^2 = 1 + \eta\Big\{-0.1875f_1 - 0.078125f_2 + 0.32142858f_1^2 + 0.24702380f_1f_2 + 0.0562500f_2^2\Big\}$$

$$+\eta^2\Big\{+0.0087890625f_1^2 + 0.0073242187 5f_1f_2 + 0.0015258789 1f_2^2$$
$$- 0.03013393f_1^3 - 0.03571429f_1^2f_2 - 0.01492280f_1f_2^2 - 0.00219727f_2^3$$
$$+ 0.02583908f_1^4 + 0.03970025f_1^3f_2 + 0.02429537f_1^2f_2^2 + 0.00694754f_1f_2^3$$
$$+ 0.000791016f_2^4\Big\}$$

X 0.35 !!!

$$\eta^3 q_0 \Big[0.175A + 0.140B + 0.05833333C + 0.24D + 0.01458333E + 0.2F$$
$$+ 0.035G + \overset{0.0696969697}{\cancel{0.1696969696}}H + 0.00583333I - 0.1J - 0.05K \Big]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
0.175		−0.35		
	0.14		−0.56	
−0.13125	−0.0984375	+0.525	+0.39375	−0.2625
			+0.54	+0.405
	−0.04375	−0.2953125	−0.26796875	−0.16611328
				+0.2
+0.021875	+0.02953125	−0.0875	−0.354375	−0.1771875
			−0.04356060 6	−0.05880682
		+0.13125	+0.275625	+0.080390625
		−0.01953125	−0.052734375	−0.035595703125

$$(+0.065625 f_1 + 0.02734375 f_2 - 0.09609375 f_1^2 - 0.06926374 f_1 f_2 - 0.01481267 f_2^2)\eta$$

$$\eta q_0 = 1 + \eta \{ -0.09375 f_1 - 0.0390625 f_2 + 0.16071429 f_1^2 + 0.12351190 f_1 f_2 + 0.02812500 f_2^2 \}$$

$$= \eta \{ 0.065625 f_1 + 0.02734375 f_2 - 0.09609375 f_1^2 - 0.06926374 f_1 f_2 - 0.01481267 f_2^2 \}$$

$$- \eta^2 \{ 0.00615234 f_1^2 + 0.00512695 f_1 f_2 + 0.00106812 f_2^2 - 0.01955566 f_1^3 -$$
$$\boxed{\begin{matrix} -0.02274714 \\ -0.01631860 f_1^2 f_2 \end{matrix}} - 0.00931728 f_1 f_2^2 - 0.00134766 f_2^3 + 0.01544364 f_1^4 +$$
$$+ 0.02300039 f_1^3 f_2 + 0.01363814 f_1^2 f_2^2 + 0.00377758 f_1 f_2^3 + 0.00041661 f_2^4 \}$$

$$\eta^{2}A\Big[0.04270833A + 0.0778571B + 0.03541667C + 0.1552381D + 0.00989583 3E$$ 405

$$\overset{0.0477689}{} $$

$$+0.14090909F + 0.02541667G + 0.1263403H + 0.00443452I - 0.02857143J$$

$$-0.04027777K\Big]$$

f_1	f_2	f_1^2	f_1f_2	f_2^2
+0.04270833		−0.08541667		
	+0.0778571		−0.3114284	
−0.0796875	−0.059765625	+0.31875	+0.2390625	−0.159375
			+0.3492857	+0.2619643
	−0.0296875	−0.200390625	−0.1818359	−0.1122197
				+0.1409091
+0.0158854	+0.0214453	−0.0635417	−0.25734375	−0.1286719
			−0.0298556	−0.0403050
		+0.0997767 +0.0997317	+0.2095311 +0.2094366	+0.0611132 +0.0610856
		−0.0157335	−0.0424805	−0.0286743

$$\left(-0.0210938f_1 + 0.0098493f_2 + 0.0533992f_1^2 - 0.0251594f_1f_2 - 0.0057869f_2^2\right)\eta$$

$$+0.0534442 \qquad -0.0250649 \qquad -0.0057587$$

$$-\left(-f_1 + 2f_1^2\right)\eta^2$$

$$= -\eta^2\Big\{+0.0210938f_1^2 - 0.0098493f_1f_2 \overset{-0.0956318}{-0.0955868}f_1^3 \overset{+0.0447635}{+0.0448580}f_1^2f_2$$

$$+\underset{+0.0057587}{0.0057869}f_1f_2^2 + \underset{+0.1068884}{0.1067984}f_1^4 - \underset{-0.0501298}{0.0503188}f_1^3f_2 - \underset{-0.0115174}{0.0115738}f_1^2f_2^2\Big\}$$

$\eta^2 B[0.0365 B + 0.03388889 C + 0.150857143 D + 0.009734848 E + 0.14 F$ <u>406</u>

$+ 0.0254615 G + \overset{0.0469417}{\cancel{0.1274459}} H + 0.0045 I - \cancel{0.0805042} J - 0.0415789 K]$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
	+0.0365		−0.146	
−0.07625	−0.0571875	+0.305	+0.22875	−0.1525
			+0.3394286	+0.2545714
	−0.0292045	−0.1971307	−0.1788778	−0.1108860
				+0.14
+0.0159134	+0.0214831	−0.0636538	−0.2577977	−0.1288988
			−0.0293386	−0.0396071
		+0.10125	+0.212625	+0.062015625
		−0.0162418	−0.0438527	−0.0296006

$-0.0603366 f_1 - 0.0284089 f_2 + 0.1292237 f_1^2 + 0.1249368 f_1 f_2 - 0.0049055 f_2^2$

$0 \quad + \quad f_2 \quad + \quad 0 \quad - 4 \quad f_1 f_2 \quad + \quad 0$

$$\eta^2 \{ -0.0603366 f_1 f_2 - 0.0284089 f_2^2 + 0.3705701 f_1^2 f_2 + 0.2385724 f_1 f_2^2$$
$$- 0.0049055 f_2^3 - 0.5168948 f_1^3 f_2 - 0.4997472 f_1^2 f_2^2 + 0.0196220 f_1 f_2^3 \}$$

$$C\left[0.00798611 C + 0.07194805 D + 0.0046875 E + 0.067948718 F + 0.0124405 G + 0.0225468\overset{3}{9} H + 0.00222222 I - 0.020833333 K\right]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
-0.0179687	-0.0134766	$+0.071875$	$+0.05390625$	-0.0359375
			$+0.1618831$	$+0.1214123$
	-0.0140625	-0.0949219	-0.0861328	-0.0533936
				$+0.0679487$
$+0.0077753$	$+0.0104967$	-0.0311013	-0.1259601	-0.0629800
			-0.0140918	-0.0190239
		$+0.05$	$+0.105$	$+0.030625$
		-0.0081380	-0.0219726	-0.0148315
$-0.0101934 f_1$	$-0.0170424 f_2$	$-0.0122862 f_1^2$	$+0.0726390 f_1 f_2$	$+0.0338195 f_2^2$
$-2.25 f_1$	$-1.6875 f_2$	$+9 f_1^2$	$+6.25 f_1 f_2$	$-4.5 f_2^2$

$$= \eta^2 \{ 0.0229352 f_1^2 + 0.0555468 f_1 f_2 + 0.0287591 f_2^2$$

$$- 0.0640967 f_1^3 \overset{-0.3643761}{-0.0995788} f_1^2 f_2 - 0.2678263 f_1 f_2^2 + 0.0196204 f_2^3$$

$$- 0.1105758 f_1^4 + 0.5707562 f_1^3 f_2 + 0.8499294 f_1^2 f_2^2 - 0.0985624 f_1 f_2^3$$

$$- 0.1521878 f_2^4 \}$$

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$$D[0.1635918\,D + 0.02148351\,6E + 0.3134694\,F + 0.05771429\,G + 0.1029900\,H + 0.0104033\,I - 0.00987755\,K]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
			+0.3680816	+0.2760612
	−0.0644505	−0.4350412	−0.3942596	−0.2447107
				+0.3134694
+0.0360714	+0.0486964	−0.1442857	−0.5843572	−0.2921786
			−0.0643688	−0.0868978
		+0.234075	+0.4915575	+0.1433709
		−0.0038584	−0.0104177	−0.0070320

$$+0.0360714\,f_1 - 0.0157541\,f_2 - 0.3491103\,f_1^2 - 0.1942642\,f_1 f_2 + 0.1020824\,f_2^2$$

$$+\,2.25\quad f_1 f_2 + 1.6625\quad f_2^2$$

$$\eta^2 \left\{ +0.0811607\,f_1^2 f_2 + 0.0254238\,f_1 f_2^2 - 0.0265850\,f_2^3 - 0.7854982\,f_1^3 f_2 - 1.0262181\,f_1^2 f_2^2 - 0.0981354\,f_1 f_2^3 + 0.1722641\,f_2^4 \right\}$$

$$E\Big[\,0.00070995\,E + 0.0208333\,F + 0.0038541667\,G$$
$$+ 0.0067805\,H + 0.00070023\,I - 0.00672348\,K\,\Big]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
	−0.0021299	−0.0143765	−0.0130453	−0.00808677
				+0.0208333
+0.0024089	+0.0032520	−0.0096354	−0.0390234	−0.0195117
			−0.0042378	−0.0057210
		+0.0157552	+0.0330859	+0.009650945 +0.0097131
		−0.0026264	−0.0070912	−0.0047865
$+0.0024089 f_1$	$+0.0011221 f_2$	$-0.0108831 f_1^2$	$-0.0303118 f_1 f_2$	−0.0076226 $-0.0075596 f_2^2$

$$-3\,f_2 - 20.25\,f_1^2 - 18.375\,f_1 f_2 - 11.390625\,f_2^2$$

$$\eta^2\Big\{-0.0072267\,f_1 f_2 - 0.0033663\,f_2^2$$

+0.0100864

$$- 0.0487802\,f_1^3 - 0.0343368\,f_1^2 f_2 + 0.0428779\,f_1 f_2^2 + 0.0098974\,f_2^3$$

+0.8353022 +0.4853356

$$+ 0.2203828\,f_1^4 + 0.8137909\,f_1^3 f_2 + 0.8340265\,f_1^2 f_2^2 + 0.4841780\,f_1 f_2^3$$

$$+ 0.0861086\,f_2^4\Big\}$$

+0.0868262

$$F\left[\,\overset{0.09907239}{0.1535714\,F + 0.0570588\,G + 0.101092626\,H}\right.$$
$$\left.+\,0.010438597\,I - 0.101242236\,K\,\right]$$

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
				+0.1535714
+0.0356618	+0.0481434	−0.1426470	−0.5777204	−0.2888602
			−0.0619202 −0.0631829	−0.0835923 −0.0852969
		+0.2348684	+0.4932237	+0.1438569
		−0.0395477	−0.1067789	−0.0720758

$$+0.0356618\,f_1 f_2^2 + 0.0481434\,f_2^3 + 0.0526737\,f_1^2 f_2^2 - 0.2544585\,f_1 f_2^3 - 0.1488646\,f_2^4$$

$$-0.2531958 \qquad -0.14710000$$

$$G[\,0.00531944\,G + 0.01824903\,H + 0.00195833\,I - 0.01916667\,K\,]$$ 412

f_1	f_2	f_1^2	$f_1 f_2$	f_2^2
+0.0033247	+0.0044883	−0.0132986	−0.0538593	−0.0269297
			−0.0114056	−0.0153976
		+0.0440625	+0.09253125	+0.0269828
		−0.0074870	−0.0202148	−0.0136450

$$+0.0033247\,f_1 + 0.0044883\,f_2 + 0.0232769\,f_1^2 + 0.0070516\,f_1 f_2 - 0.0289840\,f_2^2$$

$$+0.625\,f_1 + 0.84375\,f_2 - 2.5\,f_1^2 - 10.125\,f_1 f_2 - 5.0625\,f_2^2$$

$$\eta^2 \Big\{ 0.0020779\,f_1^2 + 0.0056104\,f_1 f_2 + 0.0037870\,f_2^2 +$$
$$+ 0.0062363\,f_1^3 - 0.0208362\,f_1^2 f_2 - 0.0744405\,f_1 f_2^2 - 0.0471773\,f_2^3 -$$
$$- 0.0581923\,f_1^4 - 0.2533076\,f_1^3 f_2 - 0.1167768\,f_1^2 f_2^2 + 0.2577643\,f_1 f_2^3$$
$$+ 0.1467315\,f_2^4 \Big\}$$

$$H\left[0.0163730\,H + 0.00328644\,I - 0.03102659\,K\right]$$

f_1^2	$f_1 f_2$	f_2^2
	-0.0102331	-0.0138147
$+0.0739449$	$+0.1552843$	$+0.0452913$
-0.0121198	-0.0327234	-0.0220883
$+0.0618251\,f_1^2$	$+0.1123278\,f_1 f_2$	$+0.0093883\,f_2^2$

$$\times \quad 0 \qquad -0.625\,f_1 f_2 - 0.84375\,f_2^2$$

$$\eta^2\left\{-0.0386407\,f_1^3 f_2 - 0.1223698\,f_1^2 f_2^2 - 0.1006443\,f_1 f_2^3 - 0.0079214\,f_2^4\right\}$$

$$K\left[0.01845238\,K - 0.00362179\,I\right]$$

f_1^2	$f_1 f_2$	f_2^2
$+0.0072080$	$+0.0194615$	$+0.0131365$
-0.0814903	-0.1711296	-0.0499128
$-0.0742823\,f_1^2$	$-0.1516681\,f_1 f_2$	$-0.0367763\,f_2^2$

$$\times)\quad 0.390625\,f_1^2 + 1.0546875\,f_1 f_2 + 0.7119140625\,f_2^2$$

$$\eta^2\left\{-0.0290165\,f_1^4 - 0.1375900\,f_1^3 f_2 - 0.2272108\,f_1^2 f_2^2 - 0.1467622\,f_1 f_2^3 - 0.0261816\,f_2^4\right\}$$

$$0.0001\,I^2 = \eta^2\left\{0.225\,f_1^2 + 0.4725\,f_1 f_2 + 0.1378125\,f_2^2\right\}^2$$

$$= \eta^2\left\{0.050625\,f_1^4 + 0.212625\,f_1^3 f_2 + 0.285271875\,f_1^2 f_2^2 + 0.1302328\,f_1 f_2^3 + 0.0189923\,f_2^4\right\}$$

$$\eta^2\, 0.0001821338\,I^2 = \eta^2\left\{0.0922052\,f_1^4 + 0.3873620\,f_1^3 f_2 + 0.5195765\,f_1^2 f_2^2 + 0.2371979\,f_1 f_2^3 + 0.0345914\,f_2^4\right\}$$

Terms with coefficient η

f_1^2	$f_1 f_2$	f_2^2
$+0.1125000$	$+0.0864583$	$+0.0196875$
-0.0960938	-0.0692637	-0.0148127

$$\eta\left(+0.0164062\,f_1^2 + 0.0171946\,f_1 f_2 + 0.0048748\,f_2^2\right) \quad \text{from } \bar{\bar{E}}_1$$

$$\quad\left(+0.0304688\,f_1^2 + 0.0319329\,f_1 f_2 + 0.0090532\,f_2^2\right) \quad \text{from } -80$$

$$\eta\left(+0.0468750\,f_1^2 + 0.0491275\,f_1 f_2 + 0.0139280\,f_2^2\right)$$

η^2

f_1^2	$f_1 f_2$	f_2^2	f_1^3	$f_1^2 f_2$	$f_1 f_2^2$	f_2^3	
+0.0003687? −0.0053442	+0.0037879 −0.0062538	+0.0010068 −0.0014977					
+0.0030162	+0.0025635	+0.0005341	−0.0105469	−0.0125000	−0.0052230	−0.0007690	
−0.0061523	−0.0051270	−0.0010681	+0.017557	+0.0227471 +0.0163186	+0.0093173	+0.0013477	
−0.0210938	+0.0078413	0	+0.0956318 +0.0955868	−0.0447635 −0.0448580	−0.0057587 −0.0057869	0	
0	−0.0603366	−0.0214089	0	+0.3705701	+0.2385724	−0.0049055	
+0.0379352	+0.0555468	+0.0277591	−0.0410917	−0.3645761 −0.0995788	−0.2678263	+0.0176204	
0	0	0	0	+0.0811607	+0.0254238	−0.0265850	
0	−0.0072267	−0.0033663	−0.0114802	−0.0343368	+0.0428779	+0.0100864 +0.0098974	
0	0	0	0	0	+0.0356618	+0.0481434	
+0.0020779	+0.0056104	+0.0037870	+0.0012363	−0.0108362	−0.0744405	−0.0471773	
+0.00121194 $-0.0045010 f_1^2$	+0.0046676 $-0.0053741 f_1 f_2$	+0.0033158 $-0.0012608 f_2^2$	−0.0020000 $-0.0020450 f_1^3$	−0.0028347 $+0.2559396 f_1^2 f_2$	−0.0013953 $-0.0014235 f_1 f_2^2$	$-0.0002389 f_2^3$	ε_1

$+0.0000529 f_1^2 + 0.0000612 f_1 f_2 + 0.0000177 f_2^2$ $\quad f_{1\,m}$ ε_3

$+0.0012648 f_1^2 + 0.0047288 f_1 f_2 + 0.0033935 f_2^2$

η^2

f_1^4	$f_1^3 f_2$	$f_1^2 f_2^2$	$f_1 f_2^3$	f_2^4
+0.0090402	+0.0138951	+0.0085034	+0.0024316	+0.0002769
−0.0154436	−0.0230004	−0.0131381	−0.0037776	−0.0004166
−0.1068884 −0.1067984	+0.0509298 +0.0503188	+0.0115174 +0.0115738	0	0
0	−0.5168948	−0.49[illegible]	+0.0[illegible]6220	0
−0.1105758	+0.5707562	+0.[illegible]4	−0.0985624	−0.15[illegible]678
0	−0.7854982	−1.0[illegible]181	−0.0981354	+0.1722641
+0.2203828	+0.[illegible]137909	+0.8353022 +0.8340265	+0.4853356 +0.4841780	+0.086262 +0.0861086
0	0	+0.0526737	−0.2531458 −0.2544585	−0.1471000 −0.1488046
−0.0581923	−0.2533076	−0.116[illegible]	+0.2571643	+0.1467315
0	−0.0386407	−0.1[illegible]18	−0.1001443	−0.0079214
−0.0290165	−0.1375900	−0.2272108	−0.14[illegible]2	−0.0261816
+0.0[illegible]52	+0.3872620	+0.5195765	+0.2371979	+0.0345914

$+\overset{+0.0015116}{\cancel{0.0016044}} f_1^4 + \overset{+0.0809023}{0.0810913} f_1^3 f_2 + \overset{+0.2715418}{0.2703223} f_1^2 f_2^2 + 0.3012737 f_1 f_2^3 + \overset{+0.1063185}{0.1044605} f_2^4$ Jrom $\mathcal{E}_1$

$+0.00[illegible]3571 f_1^4 + 0.00[illegible]7484 f_1^3 f_2 + 0.0006044 f_1^2 f_2^2 + 0.000[illegible] f_1 f_2^3 + 0.0000315 f_2^4$ Jrom $\mathcal{E}_3$

$(+0.0018687 f_1^4 + 0.0816507 f_1^3 f_2 + 0.2721462 f_1^2 f_2^2 + 0.3014961 f_1 f_2^3 + 0.1063500 f_2^4)\,\eta^2$

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$$\frac{H}{R^3} = \frac{1}{2}\frac{\sigma^3}{E^2}\pi\left(\frac{t}{R}\right)\Big[\eta\left(0.4709575 f_1^2 + 0.5756148 f_1 f_2 + 0.1942698 f_2^2\right)\frac{1}{K^2}$$

$$- \eta^2\left(0.0468750 f_1^2 + 0.0491275 f_1 f_2 + 0.0139280 f_2^2\right)$$

$$+ \eta^3\left(0.0012648 f_1^2 + 0.0047288 f_1 f_2 + 0.0033935 f_2^2 - 0.0020000 f_1^3 - 0.0028347 f_1^2 f_2\right.$$

$$- 0.0013953 f_1 f_2^2 - 0.0002369 f_2^3 + 0.0018687 f_1^4 + 0.0816507 f_1^3 f_2 + 0.2421462 f_1^2 f_2^2$$

$$\left. + 0.3014961 f_1 f_2^3 + 0.1063500 f_2^4\right)\Big]$$

If $f_2 = 0$, the conditions for equilibrium are

$$\frac{0.4709575}{K^2} - 0.0937500\eta + \eta^2\left(0.0037944 - 0.0060000 f_1 + 0.0056061 f_1^2\right) = 0$$

$$\frac{0.9419150}{K^2} - 0.0937500\eta + \eta^2\left(0.0025296 - 0.0060000 f_1 + 0.0074748 f_1^2\right) = 0$$

This set of equations can be put into the form

$$\eta^2 + A\eta + \frac{B}{K^2} = 0$$

$$\eta^2 + C\eta + \frac{D}{K^2} = 0$$

The resultant will be

$$\begin{vmatrix} C & \frac{D}{K^2} & 0 \\ 1 & A & \frac{B}{K^2} \\ 1 & C & \frac{D}{K^2} \end{vmatrix} - \begin{vmatrix} A & \frac{B}{K^2} & 0 \\ 1 & A & \frac{B}{K^2} \\ 1 & C & \frac{D}{K^2} \end{vmatrix} = 0$$

If $f_2 = 0$, the condition for equilibrium is

$$\frac{0.9419150}{K^2} - 0.0937500\,\eta + \eta^2\left(0.0025296 - 0.0060000 f_1 + 0.0074748 f_1^2\right) = 0.$$

where $\quad \eta = \frac{E}{\sigma}\left(\frac{a}{R}\right)^2$, $\quad$ Now $\quad \frac{\delta}{R} = f_1 \cdot \frac{1}{2}\left(\frac{a}{R}\right)^2$

$$\frac{\delta}{t} = \mu = \frac{1}{2}\left(\frac{a}{R}\right)^2 \frac{f_1}{\left(\frac{t}{R}\right)}$$

or $\quad f_1 = \dfrac{\mu\left(\frac{t}{R}\right)}{\frac{1}{2}\left(\frac{a}{R}\right)^2}$

Thus $$\frac{0.9419150}{K^2} - 0.0937500\left(\frac{E}{\sigma}\right)\left(\frac{a}{R}\right)^2 + \left(\frac{E}{\sigma}\right)^2\left(\frac{a}{R}\right)^4\left(0.0025296 - 0.006000\frac{\mu\left(\frac{t}{R}\right)}{\frac{1}{2}\left(\frac{a}{R}\right)^2}\right.$$
$$\left. + 0.0074748\,\frac{\mu^2\left(\frac{t}{R}\right)^2}{\frac{1}{4}\left(\frac{a}{R}\right)^4}\right) = 0$$

$$\frac{0.9419150}{K^2} - \frac{0.0937500}{K}\,\gamma^2 + \frac{1}{K^2}\left(0.0025296\,\gamma^4 - 0.012000\,\mu\gamma^2 \right.$$
$$\left. + 0.0298992\,\mu^2\right) = 0.$$

Thus
$$0.09375\,K\gamma^2 = 0.0025296\,\gamma^4 - 0.012000\,\mu\gamma^2 + \left(0.941915 + 0.0298992\,\mu^2\right)$$

$$K = 0.026982\,\gamma^2 - 0.12800\,\mu + \frac{10.047 + 0.31892\,\mu^2}{\gamma^2}$$

$$\gamma^2 = \sqrt{\frac{10.047 + 0.31892\mu^2}{0.026982}}$$

$$K = 2\sqrt{0.026982\,(10.047 + 0.31892\,\mu^2)} - 0.12800\mu$$

$$= 0.3286\sqrt{10.047 + 0.31892\,\mu^2} - 0.12800\mu$$

$\mu = 0.5,$ $\quad K = 0.1856\sqrt{31.503 + \mu^2} - 0.12800\mu$

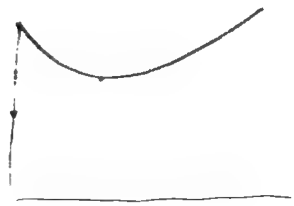
$= 1.046 -$

$\mu = 2,$ $\quad K = 0.850$

$\mu = 4,$ $\quad K = 0.768$

$\mu = 6,$ $\quad K = 0.758$

$\mu = 7,$ $\quad K = 0.767$



$$\frac{\mathcal{U}}{R^3} = \frac{1}{2}\frac{\sigma^2}{E}\pi\left(\frac{t}{R}\right)\left(\frac{a}{R}\right)^2 \Big\{ f_1^2\left(0.4709575 + 0.5756148\,\rho + 0.1942698\,\rho^2\right)\frac{1}{K^2}$$

$$- \eta f_1^2\left(0.0468750 + 0.0491275\,\rho + 0.0139280\,\rho^2\right) + \eta^2 f_1^2\Big[0.0012648 + 0.0047288\,\rho + 0.0033935\,\rho^2$$

$$- f_1\left(0.0020000 + 0.0028347\,\rho + 0.0013953\,\rho^2 + 0.0002389\,\rho^3\right) + f_1^2\big(0.0018687 + 0.0816507\,\rho + 0.2721462\,\rho^2$$

$$+ 0.3014961\,\rho^3 + 0.1063500\,\rho^4\big)\Big]\Big\}$$

$$\rho = \frac{f_2}{f_1}$$

$$\left(0.9419150 + 1.1512296\,\rho + 0.3885396\,\rho^2\right)\frac{1}{K^2} - \eta\left(0.0937500 + 0.0982550\,\rho + 0.0278560\,\rho^2\right)$$

$$+ \eta^2\Big[0.0025296 + 0.0094576\,\rho + 0.0067870\,\rho^2 - f_1\left(0.0060000 + 0.0085041\,\rho + 0.0041859\,\rho^2 + 0.0007167\,\rho^3\right)$$

$$+ f_1^2\left(0.0074748 + 0.3266028\,\rho + 1.0885848\,\rho^2 + 1.2059844\,\rho^3 + 0.425400\,\rho^4\right)\Big] = 0.$$

$$\left(0.5756148 + 0.3885396\,\rho^2\right)\frac{1}{K^2} - \eta\left(0.0491275 + 0.0278560\,\rho\right)$$

$$+ \eta^2\Big[0.0047288 + 0.0067870\,\rho - f_1\left(0.0028347 + 0.0027906\,\rho + 0.0007167\,\rho^2\right)$$

$$+ f_1^2\left(0.0816507 + 0.5442924\,\rho + 0.9044883\,\rho^2 + 0.425400\,\rho^3\right)\Big] = 0$$

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$$\frac{1}{K^2}(0.9419150 + 0.5756148\rho) - \frac{1}{K}\gamma^2(0.093750 + 0.0491275\rho)$$

$$+ \frac{1}{K^2}\gamma^4\Big[0.0025296 + 0.0047288\rho - \frac{\mu_1}{\gamma^2}(0.0120000 + 0.0113388\rho + 0.0027906\rho^2)$$

$$+ \frac{\mu_1^2}{\gamma^4}(0.0298992 + 0.9478084\rho + 2.1771696\rho^2 + 1.2059844\rho^3)\Big] = 0$$

$$\mu_1 = \frac{1}{2}\frac{\left(\frac{\rho}{R}\right)^2}{\left(\frac{t}{R}\right)} f_1 = \frac{\delta_1}{t}$$

$$\frac{1}{K^2}(0.5756148 + 0.3885396\rho) - \frac{1}{K}\gamma^2(0.0491275 + 0.0278565\rho)$$

$$+ \frac{1}{K^2}\gamma^4\Big[0.0047288 + 0.0067870\rho - \frac{\mu_1}{\gamma^2}(0.0056694 + 0.0055812\rho + 0.0014334\rho^2)$$

$$+ \frac{\mu_1^2}{\gamma^4}(0.3266028 + 2.1771696\rho + 3.6179532\rho^2 + 1.701600\rho^3)\Big] = 0.$$

$$(0.0298992 + 0.9478084\rho + 2.1771696\rho^2 + 1.2059844\rho^3)\mu_1^2 - \gamma^2(0.012 + 0.0113388\rho + 0.0027906\rho^2)\mu_1$$

$$+ \left[0.941915 + 0.5756148\rho + (0.0025296 + 0.0047288\rho)\gamma^4 - K\gamma^2(0.093750 + 0.0491275\rho)\right] = 0.$$

$$(0.3266028 + 2.1771696\rho + 3.6179532\rho^2 + 1.701600\rho^3)\mu_1^2 - \gamma^2(0.0056694 + 0.0055812\rho + 0.0014334\rho^2)\mu_1$$

$$+ \left[0.5756148 + 0.3885396\rho + (0.0047288 + 0.0067870\rho)\gamma^4 - K\gamma^2(0.0491275 + 0.0278565\rho)\right] = 0$$

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$$\mu = \left(\frac{g}{t}\right) = 0.375\,\mu_1 + 0.15625\,\mu_0 = \mu_1\,(0.375 + 0.15625\,\varrho)$$

$$\frac{\mu/\mu_1 - 0.375}{0.15625} = \varrho = 6.4000\,\mu/\mu_1 - 2.08000$$

$$\mu_1^3\Big[0.0298992 + 0.9798084\,(6.4000\,\mu/\mu_1 - 2.08000) + 2.1771696\,(40.96\,\mu^2/\mu_1^2 - 26.624\,\mu/\mu_1 + 4.3264)$$
$$+ 1.2059844\,(262.144\,\mu^3/\mu_1^3 - 255.5904\,\mu^2/\mu_1^2 + 83.06688\,\mu/\mu_1 - 8.998912)\Big]$$

$$= \left[-3.4413432\,\mu_1^3 + 48.483172\,\mu_1^2\mu - 219.06117\,\mu_1\mu^2 + 316.14157\,\mu^3\right]$$

$$\mu_1^2\Big[0.012 + 0.0113388\,(6.4000\,\mu/\mu_1 - 2.08000) + 0.0022\text{[illegible]}6\,(40.96\,\mu^2/\mu_1^2 - 26.624\,\mu/\mu_1 + 4.3264)\Big]$$

$$= \left[0.0004885\,\mu_1^2 - 0.0017286\,\mu_1\mu + 0.1143030\,\mu^2\right]$$

$$\mu_1\left[0.941915 + 0.578148\,(6.400\,\mu/\mu_1 - 2.08)\right] = -0.2553638\,\mu_1 + 3.6839347\,\mu$$

$$\mu_1\left[0.0025296 + 0.0047288\,(6.400\,\mu/\mu_1 - 2.08)\right] = -0.0073063\,\mu_1 + 0.0302643\,\mu$$

$$\mu_1\left[0.93750 + 0.0491275\,(6.400\,\mu/\mu_1 - 2.08)\right] = -0.0084352\,\mu_1 + 0.3144160\,\mu.$$

$$\mu_1^3\left[0.3266028 + 2.1771696\left(6.400\frac{\mu}{\mu_1} - 2.08\right) + 3.6179532\left(40.96\frac{\mu^2}{\mu_1^2} - 26.624\frac{\mu}{\mu_1} + 4.3264\right)\right.$$ $\underline{\underline{423}}$

$$\left. + 1.701600\left(262.144\frac{\mu^3}{\mu_1^3} - 255.5904\frac{\mu^2}{\mu_1^2} + 83.06688\frac{\mu}{\mu_1} - 8.998912\right)\right]$$

$$= -3.8617459\,\mu_1^3 + 58.9561025\,\mu_1^2\mu - 286.72126\,\mu_1\mu^2 + 446.06423\,\mu^3$$

$$\mu_1^2\left[0.0056694 + 0.0055812\left(6.400\frac{\mu}{\mu_1} - 2.08\right) + 0.0014334\left(40.96\frac{\mu^2}{\mu_1^2} - 26.624\frac{\mu}{\mu_1} + 4.3264\right)\right]$$

$$= \left[0.0002620\,\mu_1^2 - 0.0024432\,\mu_1\mu + 0.0587121\,\mu^2\right]$$

$$\mu_1\left[0.5756148 + 0.3885396\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.2325476\,\mu_1 + 2.4866534\,\mu$$

$$\mu_1\left[0.0047288 + 0.0067870\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.0093882\,\mu_1 + 0.0434368\,\mu$$

$$\mu_1\left[0.0491275 + 0.027856\left(6.4\frac{\mu}{\mu_1} - 2.08\right)\right] = -0.0088130\,\mu_1 + 0.1782784\,\mu$$

$$A_1\gamma^4 + B_1\gamma^2 + C_1 = 0$$

$$A_2\gamma^4 + B_2\gamma^2 + C_2 = 0$$

$$C_1 = +3.4413432\mu_1^3 - 48.483172\mu_1^2\mu + 219.06117\mu_1\mu^2 - 316.14157\mu^3 + 0.2553638\mu_1 - 3.6839347\mu$$

$$B_1 = \left\{0.0004885\mu_1^2 - 0.0017226\mu_1\mu + 0.1143030\mu^2 - K(0.0084353\mu_1 - 0.3144160\mu)\right\}$$

$$A_1 = (0.0043063\mu_1 - 0.0302643\mu)$$

$$C_2 = +3.8612759\mu_1^3 - 58.9561025\mu_1^2\mu + 286.721126\mu_1\mu^2 - 446.06423\mu^3 + 0.2325426\mu_1 - 2.4866534\mu$$

$$B_2 = \left\{0.0002620\mu_1^2 - 0.0074432\mu_1\mu + 0.0587121\mu^2 - K(0.0088130\mu_1 - 0.1782787\mu)\right\}$$

$$A_2 = (0.0093888\mu_1 - 0.0434368\mu)$$

The resultant for elimination of j^2 is

$$(A_1C_2 - A_2C_1)^2 - (A_1B_2 - A_2B_1)(B_1C_2 - B_2C_1) = 0$$

$\mu = 0$

$$A_1 = 0.0073063\,\mu_1 \qquad A_2 = 0.0093882\,\mu_1$$

$$B_1 = 0.0004885\,\mu_1^2 - 0.0084352\,\mu_1 K \qquad B_2 = 0.0002620\,\mu_1^2 - 0.0088130\,\mu_1 K$$

$$C_1 = 3.4413432\,\mu_1^3 + 0.2553638\,\mu_1 \qquad C_2 = 3.8617459\,\mu_1^3 + 0.2325426\,\mu_1$$

$$(A_1C_2 - A_2C_1)^2 = \mu_1^4\left(-0.0040929\,\mu_1 - 0.0069834\right)^2$$

$$(A_1B_2 - A_2B_1) = \mu_1^2\left(-0.0000026719\,\mu_1 - 0.0000148009\,K\right)$$

$$(B_1C_2 - B_2C_1) = \mu_1^2\left[0.0009848\,\mu_1^3 + 0.00004669\,\mu_1 - (0.0022460\,\mu_1^2 - 0.00028894)K\right]$$

$$(40.929\,\mu_1 + 6.9834)^2 + (0.26719\,\mu_1 + 1.48009\,K)\times$$
$$\times\left[0.9848\,\mu_1^3 + 0.04669\,\mu_1 - (2.2460\,\mu_1^2 - 0.28894\,K\right] = 0$$

$$1675.183\,\mu_1^2 + 571.647\,\mu_1 + 48.7679$$

Let $\xi = \frac{\mu_2}{\mu}$

$$A_1 = 0.0073063\,\xi - 0.0302643$$

$$B_1 = \mu\,(0.0004885\,\xi^2 - 0.0017286\,\xi + 0.1143030) - K\,(0.0084352\,\xi - 0.3144160)$$

$$C_1 = \mu^2\,(3.4413432\,\xi^3 - 48.483172\,\xi^2 + 219.06117\,\xi - 316.14157) + (0.2553638\,\xi - 3.6839147)$$

$$A_2 = 0.0093882\,\xi - 0.0434368$$

$$B_2 = \mu\,(0.0002620\,\xi^2 - 0.0024432\,\xi + 0.0587121) - K\,(0.0088130\,\xi - 0.1782784)$$

$$C_2 = \mu^2\,(3.8617459\,\xi^3 - 58.9561025\,\xi^2 + 286.72126\,\xi - 446.06423) + (0.2325476\,\xi - 2.4866534)$$

Let $\mu = 7$

$$A_1 = 0.0073063\,\xi - 0.0302643$$

$$B_1 = 0.0034195\,\xi^2 - 0.0121002\,\xi + 0.800121 - K\,(0.0084352\,\xi - 0.3144160)$$

$$C_1 = 168.62582\,\xi^3 - 2375.6754\,\xi^2 + 10733.99733\,\xi - 15490.93693 + 0.2553638\,\xi - 3.6839147$$

$$C_1 = 168.62582\,\xi^3 - 2375.6754\,\xi^2 + 10734.252\,\xi - 15494.621$$

$$A_2 = 0.0093882\,\xi - 0.0434368$$

$$B_2 = 0.001834\,\xi^2 - 0.0171024\,\xi + 0.4109847 - K\,(0.0088130\,\xi - 0.1782784)$$

$$C_2 = 189.22555\,\xi^3 - 2888.8490\,\xi^2 + 14049.575\,\xi - 21859.634$$

We put

$$A_1 = a_1 \qquad A_2 = a_2$$

$$B_1 = b_1 - c_1 K \qquad B_2 = b_2 - c_2 K$$

$$C_1 = d_1 \qquad C_2 = d_2$$

$$\therefore \quad (a_1 d_2 - a_2 d_1)^2 - \left[a_1 b_2 - a_2 b_1 - K(a_1 c_2 - a_2 c_1)\right]\left[b_1 d_2 - b_2 d_1 - K(c_1 d_2 - c_2 d_1)\right] = 0$$

$$\therefore \left\{(a_1 d_2 - a_2 d_1)^2 - (a_1 b_2 - a_2 b_1)(b_1 d_2 - b_2 d_1)\right\}$$

$$+ \left\{(a_1 c_2 - a_2 c_1)(b_1 d_2 - b_2 d_1) + (a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)\right\} K$$

$$+ (a_1 c_2 - a_2 c_1)(c_1 d_2 - c_2 d_1) K^2 = 0$$

	$\xi = -10$		0		$\xi = +10$	
	1	2	1	2	1	2
a	−0.1033273	−0.1373188	−0.0307643	−0.0434368	+0.0427987	+0.0504452
b	+1.263073	+0.7654087	+0.800121	+0.4159847	+1.0210690	+0.4233607
c	−0.398768	−0.2664084	−0.3144160	−0.1282284	−0.2300640	−0.0901484
d	−529030.50	−640465.83	−15494.621	−21857.654	+22706.18	+18926.77
$a_1d_2-a_2d_1$	−6468.228	1	−11.4702	1	−343.326	1
$a_1b_2-a_2b_1$	+0.0943561	2	+0.0223165	2	−0.0333888	2
$b_1d_2-b_2d_1$	−404030.71	3	−11122.300	3	+9679.015	3
$a_1c_2-a_2c_1$	−0.0272311	4	−0.00826175	4	+0.0077474	4
$c_1d_2-c_2d_1$	+114459.11	5	+4110.6624	5	−2300.916	5
$(1)^2-(2)(3)$	41876096	6	379.776	6	118195.9	6
$(3)(4)+(2)(5)$	21802.116	7	183.625	7	151.812	7
$(4)(5)$	−3116.847	8	−33.9613	8	−17.8261	8
$(7)/(8)$	−6.99493	9	−5.40689	9	−8.51628	9
$(6)/(8)$	−13435.403	10	−11.18261	10	−6630.497	10
$-(9)/2$			+2.70345	11	+4.25814	11
$(11)^2-(10)$			+18.49125	12	+6648.63	12
$(12)^{\frac{1}{2}}$			+4.30015	13	+81.5391	13
$(11)+(13)$			+7.00360	K	+85.79726	K

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	function f_1	function f_2	function f_3	$0.625\, f_3$	$f_1 - 0.625 f_3$	W_1	$0.84375\, f_3$	$f_2 - 0.84375 f_3$	W_2		
0	1.0000	1.00000	1.00000	0.6250000	0.3750000	1.000000	0.84375000	0.15625000	1.000000		
0.1	0.98010	0.997001	0.99980001	0.62487501	0.35522499	0.947267	0.84358126	0.15441974	0.988286		
0.2	0.92160	0.984064	0.99680256	0.62300160	+0.29859840	0.796262	0.84105216	0.14301184	0.915276		
0.3	0.82810	0.946729	0.98386561	0.61491601	+0.21318399	0.568491	0.83013661	0.11659239	0.746191		
0.4	0.70560	0.876096	0.94945536	0.59340960	+0.11219040	0.299174	0.80110296	0.07499304	0.479955		
0.5	0.56250	0.765625	0.87890625	0.54931641	+0.01318359	0.035156	0.74157715	0.02404785	0.153906		
0.6	0.40960	0.614656	0.75759616	0.47349760	-0.06389760	-0.170394	0.63922176	-0.0245686	-0.157221		
0.7	0.26010	0.431649	0.57744801	0.36090501	-0.10080501	-0.268813	0.48722176	-0.05557276	-0.355666		
0.8	0.12960	0.238144	0.34857216	0.21785760	-0.08825760	-0.235354	0.29410776	-0.05596376	-0.358168		
0.9	0.03610	0.073441	0.11826721	0.07391701	-0.03781701	-0.100845	0.09978796	-0.02634696	-0.168621		

The difficulty now is evidently that the wave form W_1, W_2 will not give a single-wave buckle.

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With $\nu=0$, the equations on p. 383 become

$$q_2 + 0.666667\, s_2 - 0.083333 = \eta\left\{0.333333\, p_2 + 0 + 0.00173611 f_1 - 0.00[illegible] f_2 - 0.00138889 f_3\right\}$$

$$q_2 + 0 - 0.0833333 = \eta\left\{0.333333\, p_2 + 3 r_2 - 0.02604167 f_1 + 0.01845238 f_2 + 0.02083333 f_3\right\}$$

$$q_2 + 0.333333\, s_2 + 0.0833333 = \eta\left\{-0.33333\, p_2 - r_2 + 0.00868056 f_1 - 0.01145575 f_2 - 0.0111111 f_3\right\}$$

$$q_2 + 2 s_2 + 0.250000 = \eta\left\{-p_2 + 0 - 0.06770833 f_1 - 0.07946429 f_2 - 0.08750000 f_3\right\}$$

$$q_2 + 0 - 0.250000 = \eta\left\{+p_2 + 3 r_2 - 0.00520833 f_1 + 0.06741071 f_2 + 0.0750000 f_3\right\}$$

$$0.66666667\, s_2 + 0 = \eta\left\{0 - 9 r_2 + 0.02777778 f_1 - 0.01875000 f_2 - 0.0[illegible]222222 f_3\right\}$$

$$0.3333333\, s_2 + 0.16666667 = \eta\left\{-0.6666667\, p_2 - 3 r_2 + 0.0347222 f_1 - 0.03020833 f_2 - 0.03194444 f_3\right\}$$

$$1.66666667\, s_2 + 0.16666667 = \eta\left\{-0.666667\, p_2 + r_2 - 0.07638889 f_1 - 0.06770834 f_2 - 0.07638889 f_3\right\}$$

$$2\, s_2 + 0.500000 = \eta\left\{-2 p_2 - 3 r_2 - 0.06250000 f_1 - 0.14687500 f_2 - 0.16250000 f_3\right\}$$

$$s_2 + 0 = \eta\left\{0 - 3 r_2 + 0.04166667 f_1 - 0.02812500 f_2 - 0.0333333 f_3\right\}$$

$$s_2 + 0.50000 = \eta\left\{-9 p_2 - 9 [illegible] f_1 - 0.09062500 f_2 - 0.09583333 f_3\right\}$$

$$s_2 + 0.10000 = \eta\left\{-0.4 p_2 + 0.6 r_2 - 0.04583333 f_1 - 0.04062500 f_2 - 0.04583333 f_3\right\}$$

$$s_2 + 0.25000 = \eta\left\{-p_2 - 1.5 r_2 - 0.03125000 f_1 - 0.07343750 f_2 - 0.08125000 f_3\right\}$$

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$$0.5000000 = \eta\left\{-2p_2 - 6r_2 + 0.06250000f_1 - 0.06250000f_2 - 0.06250000f_3\right\}$$

$$0.400000 = \eta\left\{-1.6p_2 - 9.6r_2 + 0.15000000f_1 - 0.05000000f_2 - 0.05000000f_3\right\}$$

$$0.150000 = \eta\left\{-0.6p_2 - 2.1r_2 + 0.01458333f_1 - 0.03281250f_2 - 0.03541667f_3\right\}$$

$$0.2500000 = \eta\left\{-p_2 - 3r_2 + 0.03125000f_1 - 0.03125000f_2 - 0.03125000f_3\right\}$$

$$0.250000 = \eta\left\{-p_2 - 6r_2 + 0.09375000f_1 - 0.03125000f_2 - 0.03125000f_3\right\}$$

$$0.250000 = \eta\left\{-p_2 - 3.5r_2 + 0.02430556f_1 - 0.05468750f_2 - 0.05902777778f_3\right\}$$

$$3r_2 = 0.06250000f_1 + 0 + 0$$

$$2.5r_2 = 0.06944444f_1 + 0.02343750f_2 + 0.02777778f_3$$

$$3r_2 = 0.06250000f_1$$

$$3r_2 = 0.08333333f_1 + 0.02812500f_2 + 0.03333333f_3$$

$$0 = 0.02083333f_1 + 0.02812500f_2 + 0.03333333f_3$$

$$\boxed{-f_3 = 0.625000f_1 + 0.843750f_2}$$

Same thing !!!

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